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Oscillatory Solutions of the Equation* $y'(x) = m(x)y(x - n(x))$

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I. INTRODUCTION AND NOTATION

In this paper we shall consider the asymptotic behavior of oscillatory solutions of the equation

$$y'(x) = m(x)\,y(x - n(x)) \tag{1.1}$$

where m, n, y are real valued functions of a real variable x and "$'$" denotes the right hand derivative with respect to x. We shall also assume that n and m are defined for $x \geqslant 0$, and, with the possible exception of a finite number of points in any closed interval, are continuous. We further asssume that there exists a $\Delta > 0$ such that $0 \leqslant n(x) \leqslant \Delta$ for $x \geqslant 0$. The case in which m satisfies the bound $0 \leqslant |m(x)| \leqslant 1$ for all $x \geqslant 0$ shall be referred to as the equation of mixed type. The case in which we have $-1 \leqslant m(x) \leqslant 0$ for all $x \geqslant 0$, will be referred to as the stable equation. The case in which we have $0 \leqslant m(x) \leqslant 1$ for all $x \geqslant 0$, will be referred to as the unstable equation. We note that had we set $m_0 = \text{l.u.b.}_{x \leqslant 0} |m(x)|$ then this bound could be altered by a change of the independent variable of the form $u = kx$. However, the product $m_0\Delta$ is not altered by such a change of variables. Thus we have chosen to set $m_0 = 1$ and consider Δ as a parameter.

We shall also consider the equation

$$y'(x) = \int_0^{\infty} y(x - s)\, d\alpha(x, s) \tag{1.2}$$

where for a fixed $x \geqslant 0$, $\alpha(x, s)$ is of bounded variation in s, $\alpha(x, 0) = 0$, and

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$\alpha(x, \Delta) = \alpha(x, s)$ for $s \geqslant \Delta$ (where $\int$ denotes a R.S. integral). A point x_0 is said to be a point of mean continuity for $\alpha(x, s)$ if

$$\lim_{t \to x_0} \int_0^\infty | \alpha(x_0, s) - \alpha(t, s)| \, ds = 0.$$

We shall restrict ourselves to the case in which $\alpha(x, s)$, with the possible exception of a finite number of points in any closed interval, is continuous in the mean for all $x \geqslant 0$. If for every $x \geqslant 0$, the total variation of $\alpha(x, s)$ is bounded by 1 we say that (1.2) is a equation of mixed type. The case in which $\alpha(x, s)$ for every $x \geqslant 0$ is a nondecreasing function of s and its total variation is bounded by 1 will be referred to as the unstable equation. The case in which $\alpha(x, s)$, for every $x \geqslant 0$, is a nonincreasing function of s and its total variation is bounded by 1 will be referred to as the stable equation. Equations of the form (1.2)[(1.1)] have been studied extensively by Myskis [3] under the additional restriction that $\alpha(x, s)$ be continuous in the mean [m, n are continuous] for all $x \geqslant 0$. These special cases will be denoted by (1.2c) and (1.1c).

Let φ be a continuous function defined on $[-\Delta, 0]$ and m_1, n_1 as in equation (1.1). Then a continuous function y defined for $x \geqslant -\Delta$ will be called the solution of (1.1) associated with $[\varphi, m_1, n_1]$ if $y(x) = \varphi(x)$ for $x \in [0, -\Delta]$ and satisfies equation (1.1) at the common points of continuity of n_1 and m_1 for $x \geqslant 0$. The fact that corresponding to any such triple $[\varphi, m_1, n_1]$ there exists a unique solution y is well known [1], [3]. Here after, the statement that for a given value of Δ the equation (1.1) possesses a solution y with a given property shall mean that there exists a triple $[\varphi, m_1, n_1]$, as described above, such that $n_1(x) \leqslant \Delta$ for all $x \geqslant 0$ and the associated solution y has the given property. For equations of the form (1.2) we have similar definitions with the triple $[\varphi, m_1, n_1]$ replaced by the pair $[\varphi, \alpha_1(x, s)]$. A solution y of (1.2) will be called oscillatory if y assumes both positive and negative values for arbitrarily large values of x.

Myskis [3] has shown for the stable equation (1.2c) that if $\Delta < 3/2$ then all oscillatory solutions tend to zero as $x \to \infty$ and if $\Delta = 3/2$ then all oscillatory solutions are bounded. For $\Delta > 3/2$ one may construct unbounded oscillatory solutions of (1.1c). In Section III we show that this result is true for equations (1.2) of stable type. We shall further show that for $\Delta = 3/2$ there exists a class C of solutions of (1.1) of a very simple form such that any oscillatory solution of (1.2) which does not tend to zero is asymptotic to a solution in C. We further show that every periodic solution of the stable equation (1.2) for $\Delta = 3/2$ is in fact a solution of (1.1) but not of (1.1c). This fact, along with what appears to be a similar behavior for the unstable equation, was the reason for considering the larger set of equations (1.2).

It has been conjectured [3] for the unstable equation (1.1c) that if

$\Delta \leqslant 11/4 + \ln 2 = \Delta_c$ then all oscillatory solutions are bounded. In Sections IV, V, and VI of this paper we shall establish this conjecture for the unstable equation (1.2). In Section II we shall show that if for a given value of Δ the unstable equation (1.2) has an oscillatory solution which does not tend to zero, then for all larger values of Δ, it has an unbounded oscillatory solution. Thus for $\Delta < \Delta_c$ all oscillatory solutions of the unstable equation (1.2) tend to zero as $x \to \infty$. For $\Delta = \Delta_c$ there exists a periodic function g which is a solution of the unstable equation (1.1) but not of the equation (1.1c). The results obtained in Sections IV, V, and VI suggest that all oscillatory solutions for $\Delta = \Delta_c$ of the unstable equation (1.2) which do not tend to zero as $x \to \infty$ will be asymptotic to a scalar multiple of g. As we shall note later, this result has been established by Buchanan [2] for a special subclass of solutions of the unstable equation (1.1) but it is not known for the general case.

Finally we note that using the results of Myskis ([3], pp. 77 and 41) it is easy to show [2] that for $\Delta \leqslant 1/e$ every oscillatory solution of the equation (1.2) of mixed type must tend to zero. Again using the results of Myskis ([3], p. 67) one may construct [2] for every value of $\Delta > 1/e$ an unbounded oscillatory solution for the equation (1.1) of mixed type.

II. General Reduction Theorems

For any solution y of (1.2) let $y(\infty) = \lim_{x\to\infty} \sup |y(x)|$. In this section we shall show that if for a given value of Δ the unstable equation (1.2) possesses an oscillatory solution $y, y(\infty) \neq 0$, then for all larger values of Δ the unstable equation (1.2) possesses an oscillatory solution z, $z(\infty) = \infty$. We shall also show that if the unstable equation (1.2) possesses an oscillatory solution z, $z(\infty) = \infty$, then the unstable equation (1.1), for the same value of Δ, possesses an oscillatory solution w, $w(\infty) = \infty$ satisfying the following conditions:

(2.1) The zeros of w, for $x > 0$, form a countable set of points w_i, $w_i < w_{i+1}$ for $i = 1, 2, \ldots$, and $\lim_{i\to\infty} w_i = \infty$. The solution w changes sign at each w_i.

(2.2) $w(x) = \max[1 + x, -1]$ for $x \in [0, -\Delta]$ and $w_0 = \max[-\Delta, -1]$. Each interval $\sigma_i = [w_i, w_{i+1}]$, $i \geqslant 0$, contains a unique point x_i at which $|w(x_i)| = \max_{x\in\sigma_i} |w(x)| = M_i$ and $x_0 = 0$.

(2.3) The solution w is strictly monotone (written s.m.) on every interval $\delta_i = [x_i, x_{i+1}]$, $i \geqslant 0$.

(2.4) If $x \in \delta_i$, $i \geqslant 0$, let $\gamma(x) = \max_{0\leqslant s\leqslant\Delta} w(x - s)$ for i odd and $\gamma(x) = \min_{0\leqslant s\leqslant\Delta} w(x - s)$ for i even. If m and n are the functions associated with w by equation (1.1) then for all $x \geqslant 0$, $m(x) = 1$ and $n(x) \geqslant 0$ is the smallest number for which $w(x - n(x)) = \gamma(x)$.

We shall in Sections IV, V, and VI need a slight modification of condition (2.4) which we shall now state.

(2.4a) If $x \in \delta_i$, $i \geqslant 0$, let $\gamma(x) = \max w(x - s)$ for i odd and $\gamma(x) = \min w(x - s)$ for i even where $0 \leqslant s \leqslant \Delta$ and $x - s \in (\delta_i \cup \delta_{i-1})$. If m and n are the functions associated with w by equation (1.1) then for all $x \geqslant 0$, $m(x) = 1$ and $n(x) \geqslant 0$ is the smallest number for which $w(x - n(x)) = \gamma(-x)$.

We note that although the results of this section are stated for the unstable equation (1.2) it will be clear from the proofs that analogous results are valid for the stable equation (1.2) and the equation of mixed type, assumming of course the appropriate modification of Condition (2.4). We also note that we shall refer to any solution y of (1.2) for which $y(\infty) = \infty$ as an unbounded solution.

THEOREM 2.1. *If for $\Delta = \Delta_0$, the unstable equation* (1.2) *has an oscillatory solution w, $w(\infty) \neq 0$, then for any $\epsilon > 0$ if $\Delta = \Delta_0 + \epsilon$ the unstable equation* (1.2) *has an unbounded oscillatory solution y.*

Proof. Let $\{a_i\}$ be any sequence of zeros of $w(x)$ such that $a_{i+1} - a_i > 2\Delta_0$, $i = 0, 1, 2, \ldots$. We may assume that $a_0 = 0$. We now define $\alpha_1(x, s)$ in terms of $\alpha(x, s)$ as follows:

$$\alpha_1(x, s) = \alpha(x, s) \quad \text{for} \quad s \geqslant 0 \quad \text{and} \quad x \leqslant a_1 \quad \text{or} \quad x \in (a_i + \Delta_0, a_{i+1}) \quad i = 1, 2, \ldots,$$

$$\alpha_1(x, s) = \alpha(x, s) \quad \text{for} \quad s < x - a_i \quad \text{and} \quad x \in [a_i, a_i + \Delta_0] \quad i = 1, 2, \ldots,$$

$$\alpha_1(x, s) = (1 + \epsilon/\Delta_0)\, \alpha(x, s) \quad \text{for} \quad s \geqslant x - a_i \quad \text{and} \quad x \in [a_i, a_i + \Delta_0] \quad i = 1, 2, \ldots.$$

Let $y(x) = w(x)$ for $x \leqslant a_1$ and $y(x) = (1 + \epsilon/\Delta_0)^n\, w(x)$ for $x \in [a_n, a_{n+1}]$. It then follows from the definition of $\alpha_1(x, s)$ that y is a solution of (1.2) with the kernel $\alpha_1(x, s)$. If $\mathrm{V}_0^\infty\, \alpha(x, s)$ and $\mathrm{V}_0^\infty\, \alpha_1(x, s)$ denote the total variation of $\alpha(x, s)$ and $\alpha_1(x, s)$, respectively, for fixed x and $s \in [0, \infty]$ then we have $\mathrm{V}_0^\infty\, \alpha_1(x, s) \leqslant (1 + \epsilon/\Delta)\, \mathrm{V}_0^\infty\, \alpha(x, s)$. It is also clear that for all $x \geqslant 0$, $\alpha_1(x, s)$ is a nondecreasing function of s and $\alpha_1(x, \Delta) = \alpha_1(x, s)$ for $s \geqslant \Delta$. Finally we note that if x_0 is a point of mean continuity of $\alpha(x, s)$ then x_0 is also a point of mean continuity of $\alpha_1(x, s)$. For if $x_0 \in [a_i, a_i + \Delta]$ and $t > x_0$ we have

$$\begin{aligned}\int_0^{\Delta_0+1} | \alpha_1(x_0, s) - \alpha_1(t, s)| \, ds &\leqslant \int_0^{x_0-a_i} | \alpha(x_0, s) - \alpha(t, s)| \, ds \\ &\quad + \int_{x_0-a_i}^{t-a_i} | k\alpha(x_0, s) - \alpha(t, s)| \, ds \\ &\quad + k \int_0^{\Delta_0+1} | \alpha(x_0, s) - \alpha(t, s)| \, ds \\ &\leqslant (1 + k) \int_0^{\Delta_0+1} | \alpha(x_0, s) - \alpha(t, s)| \, ds \\ &\quad + | t - x_0 |(k + 1)\end{aligned}$$

where $k = (1 + \epsilon/\Delta_0)$. A similar argument holds for $t < x_0$. For x_0 and t in $(a_i + \Delta_0, a_{i+1})$ we have

$$\int_0^{\Delta_0+1} | \alpha_1(x_0, s) - \alpha_1(t, s)| \, ds = \int_0^{\Delta_0+1} | \alpha(x_0, s) - \alpha(t, s)| \, ds.$$

Thus we have obtained an unbounded oscillatory solution y of the equation $y'(x) = \int_0^\infty y(x - s) \, d\alpha_1(x, s)$, where $[V_0^\infty \alpha_1(x, s)] \, \Delta_0 \leqslant (1 + \epsilon/\Delta_0) \, \Delta_0 = \Delta_0 + \epsilon$. Now in view of the comments in section I the change of variable $u = (1 + \epsilon/\Delta_0) \, x$ gives us our desired result. This completes the proof of Theorem 2.1.

The proof of the fact that if for a given value of Δ, the unstable equation (1.2) possesses an unbounded oscillatory solution then unstable equation (1.1) possesses an unbounded oscillatory solution satisfying conditions (2.1)–(2.4) will be done in two steps. Thus in Theorem 2.3 we shall establish the existence of an unbounded oscillatory solution of the unstable equation (1.1) satisfying condition (2.1). Theorem 2.4 then establishes the rest of our assertion. In preparation for the proof of Theorem 2.3 we shall first state and prove an approximation lemma. The proof of this lemma, and consequently the proof of Theorem 2.3 and 2.4 only requires that a solution z of equation (1.2) be a continuous function. Thus our results are in fact valid in any class of equations of the form (1.2) where sufficient restrictions are imposed on $\alpha(x, s)$ to assure continuous solutions.

Lemma 2.2. *If for a given value of Δ, the unstable equation* (1.2) *has an unbounded oscillatory solution z, then for any $\epsilon > 0$ and $M > 0$ there exists a solution y of the unstable equation* (1.1) *defined for $x \in [-\Delta, M]$ such that $y(x) = z(x)$ for $x \in [-\Delta, 0]$ and $| y(x) - z(x)| < \epsilon$ for $x \in [0, M]$.*

Proof. Since z is continuous on $[-\Delta, M]$, for any $\epsilon_1 > 0$ there exists $\sigma_1 > 0$ such that $|z(x_1) - z(x_2)| < \epsilon_1$, for any x_1, x_2 in $[-\Delta, M]$ if $|x_1 - x_2| < \sigma_1$. We further insist that σ_1 is chosen so small that $\sigma_1 \max_{x\in[-\Delta,M]} |z(x)| < \epsilon_1$, and $(1 + \sigma_1)^{1/\sigma_1} < 2e$. Now select an integer k so large that $M/k = \sigma_2 < \sigma_1$ and define the partition points $p_i = i\sigma_2$, $i = 0,..., k$, of $[0, M]$. Divide $[-\Delta, 0]$ into subintervals of length at most σ_2 and denote the corresponding partition points by

$$0 = p_0 > p_{-1} > \cdots > p_{-h} = -\Delta.$$

We define $y(x) = z(x)$ for $x \in [-\Delta, 0]$. Now assume that $y(x)$ has been defined for $x \in [-\Delta, p_i]$ and that $a_i = \max_{j\leqslant i} |y(p_j) - z(p_j)|$. Let $p_i^+[p_i^-]$ denote any p_j such that

$$p_j \in [p_i - \Delta + \sigma_2, p_i] \qquad \text{and} \qquad z(p_j) = \operatorname{Max} z(p_n)[\min z(p_n)]$$

for $p_n \in [p_i - \Delta + \sigma_2, p_i]$. We then have that

$$\begin{aligned} y(p^+) + a_j + \epsilon_1 &\geqslant \max_{x\in[p_i-\Delta,p_{i+1}]} z(x) \geqslant [z(p_{i+1}) - z(p_i)]/\sigma_2 \\ &\geqslant \min_{x\in[p_i-\Delta,p_{i+1}]} z(x) \geqslant y(p_i^-) - a_j - \epsilon_1 . \end{aligned}$$

If $z(p_{i+1}) - z(p_i) \geqslant 0$ we set $n(x) = -p_i^+ + x$ for $x \in [p_i, p_{i+1}]$ and select $0 \leqslant m(x) \leqslant 1$ and continuous on $[p_i, p_{i+1}]$ so as to minimize a_{i+1}. If $z(p_{i+1}) - z(p_i)$ is negative we set $n(x) = x - p_i^-$ for $x \in [p_i, p_{i+1}]$ and again select $0 \leqslant m(x) \leqslant 1$ and continuous on $[p_i, p_{i+1}]$ so as to minimize a_{i+1}. Thus we obtain for a_{i+1} the estimate $a_{i+1} \leqslant a_i + (a_i + \epsilon_1)\sigma_2$. However $m(x)$ may be multivalued at p_i and in addition we want y to be s.m. on every interval $[p_i, p_{i+1}]$, $i \geqslant 0$, and $y(p_i) \neq 0$ for $i \geqslant 1$. By allowing an error term $\epsilon_2 = \epsilon_1\sigma_2$ we may assume that m is continuous for all $x \in [0, M]$ and y has the indicated properties. Thus we have

$$a_{i+1} \leqslant a_i(1 + \sigma_2) + 2\epsilon_1\sigma_2 = Aa_i + B \qquad \text{for} \qquad i \geqslant 1.$$

Since $a_0 = 0$ we have that

$$a_n \leqslant [(A^n - 1)/A - 1]\, B \qquad \text{for} \qquad i \geqslant 1.$$

Setting $n = k = M/\sigma_2$ we have that

$$a_k \leqslant ((1 + \sigma_2)^{(1/\sigma_2)M} - 1)\, 2\epsilon_1 \leqslant (2^M e^M - 1)\, 2\epsilon_1 .$$

Thus if $2^M e^M 2\epsilon_1 < \epsilon$ we will have that $|y(x) - z(x)| \leqslant a_k + 2\epsilon_1 < \epsilon$ for all $x \in [-\Delta, M]$. This completes the proof of Lemma 2.2.

We note that the solution y which was constructed above had the property that it had only a finite number of zeros in $[0, M]$ and at each of these zeros $y(x)$ changed sign.

THEOREM 2.3. *If for a given value of Δ the unstable equation* (1.2) *possesses an unbounded oscillatory solution w then, for the same value of Δ, the unstable equation* (1.1) *possesses an unbounded oscillatory solution satisfying condition* (2.1).

Proof. Since w is unbounded we may assume that after an appropriate translation and scalar multiplication we have $w(0) = 1$, $|w'(x)| \leqslant 1$, $|w(x)| \leqslant 1$, for $x \in [-\Delta, 0]$. Since $w(x)$ is an oscillatory solution, at any x_0 for which $|w(x_0)| > 0$ the interval $[x_0 - \Delta, x_0]$ must contain points x for which $w(x) = -w(x_0)/\Delta$. Thus we may assume that there exists $u > 8\Delta$ such that $w(u) < -M_1 = -\max[8\Delta + 1, 9]$. Set $M = u + 2\Delta$ and let y denote the solution of (1.1) defined on $[-\Delta, M]$ by Lemma 3.2 corresponding to $0 < \epsilon < \min[1, \max_{x \in [u, u+2\Delta]} w(x)]$. Thus we have $y(u) < -\max[8\Delta, 8]$. Let x_1 denote the first $x > u$ for which $y(x) = y(u)/2$ and x_2 the first $x > u$ for which $y(x) = 0$. Then $x_2 - u < \Delta$ and there is an $x \in [x_1, x_1 - \Delta]$ for which $y(x) > -y(u)/(2\Delta) > 4$. Let x_3 denote the last value of x before u at which $y(x) = -2$. Starting at x_3 we redefine $n(x)$, $m(x)$ for $x \in [x_3, x_1]$ as follows. Let $m(x) = 1$ and $n(x) = x - p_i$, where p_i is such that $y(p_i) = \max_{p_j \in [x, x-\Delta]} y(p_j)$. Let x_4 denote the first value of x in $[x_3, x_1]$ for which the new solution $y(x) = 2$. Then we have for $x \in [x_3, x_4]$ that $y'(x) \geqslant 4$. We now set $p_0^* = x_4$ and let $p_{-l}^* \in [x_3, x_4]$ be such that $y(p_{-l}^*) = 2y(p_{-l})$. We now redefine $n(x)$ and $m(x)$ for $x \geqslant x_4$. For all $x \geqslant x_4$ we set $m(x) = m(x - x_4)$. For $x \geqslant x_4$ if $(x - x_4) - n(x - x_4) = p_{-i}$ for $i \geqslant 1$ we set $n(x) = x - p_{-i}^*$ and if $(x - x_4) - n(x - x_4) \geqslant 0$ we set $n(x) = n(x - x_4)$. The corresponding solution y of (1.1) then for all $x \geqslant x_4$ satisfies the identity $y(x) = 2y(x - x_4)$. It is also clear from the construction that y satisfies Condition (2.1). This completes the proof of Theorem 2.3.

THEOREM 2.4. *If for a given value of Δ, the unstable* (1.2) *possesses an unbounded oscillatory solution then the unstable equation* (1.1) *has an unbounded oscillatory solution satisfying conditions* (2.1)–(2.4).

Proof. In the proof of Theorem 2.3 we may assume that if w_1 is the first positive zero of $w(x)$ then for $x \in (0, w_1)$ we have $w(x) < w(0) = 1$. Thus we shall assume that the solution y obtained by the construction given in Theorem 2.3 possesses the additional property that if y_1 denotes the first positive value of x for which $y(x) = 0$ then $y(x) < y(0)$ for $x \in (0, y_1]$. Let y_i denote the zeros of y as indicated in condition (2.1) and M_i corresponding to y be as defined in condition (2.2). For $x \in [-\Delta, 0]$ we define the function $\sigma(x) = \max[1 + x, -1]$. We then let $z(x)$ denote the solution of (1.1) corresponding to the initial function σ for $x \in [-\Delta, 0]$, and satisfying conditions (2.1)–(2.4) with the successive maxima and minima $(-1)^i M_i$. With the initial function σ and the M_i specified it is clear that the solution z of (1.1) along with its associated functions n_1 and m_1 are completely determined by conditions (2.1)–(2.4) so long as z may be continued. Thus we now show that z may be continued for all $x \geqslant 0$ and assumes all the successive extreme values $(-1)^i M_i$. Let z_i, $i \geqslant 1$ denote the positive zeros of z as defined in condition (2.1) and set $\sigma_i(z) = [z_i, z_{i+1}]$ $\sigma_i(y) = [y_i, y_{i+1}]$ for $i \geqslant 1$. Let $x_i(y)$ be such that $x_i(y) \in \sigma_i(y)$ and $y(x_i(y)) = M_i(-1)^i$. Let $x_i(z)$ be similarly defined. Let $I_{-1} = [-1, 1]$ and $I_i = [(-1)^i M_i, (-1)^{i+1} M_{i+1}]$ for $i \geqslant 0$. Then for $\xi \in I_i$ we define the function $\varphi_i(y)$ as follows

$$\varphi_i(y, \xi) = \underset{x \in \delta_i(y)}{\text{l.u.b}} [x : y(x) = \xi].$$

We now assume that z has been defined on $[-\Delta, x_{j+1}(z)]$ and that for all $\xi, \eta \in I_i$, $i \leqslant j$, we have

$$| \varphi_i(y, \xi) - \varphi_i(y, \eta)| \geqslant | \varphi_i(z, \xi) - \varphi_i(z, \eta)|,$$

and establish the result for $j + 1$. Consider the case in which $j + 1$ is odd (the argument for $j + 1$ even is similar). Then it follows from the induction hypothesis and the definition of z that $y'(x_{j+1}(y) + \sigma) \leqslant z'(x_{j+1}(z) + \sigma)$ and $y(x_{j+1}(y) + \sigma) \leqslant z(x_{j+1}(z) + \sigma)$ for all

$$\sigma \in [0, x_{j+2}(z) - x_{j+1}(z)] \subset [0, x_{j+1}(y) - x_{j+1}(y)].$$

However, the last inequality along with the definition of z and the induction hypothesis implies that $y'(\varphi_{j+1}(y, \xi)) \leqslant z'(\varphi_{j+1}(z, \xi)$— for all $\xi \in I_{j+1}$. Thus $| \varphi_{j+1}(y, \xi) - \varphi_{j+1}(y, \eta)| \geqslant | \varphi_{j+1}(z, \xi) - \varphi_{j+1}(z, \xi)|$ for all η, $\xi \in I_{j+1}$. It now follows that z may be continued to $x_i(z)$ for every i. However, $x_i(z) \geqslant \ln M_i$ and since for every integer m there exists an i such that $M_i > m$ we have that z is defined for all $x \geqslant 0$. It is also clear that every interval $[0, M]$, M any integer, can contain only a finite number of the points $x_i(z)$. Thus n_1 is continuous in $[0, M]$ with the possible exception of a finite number of points. This completes the proof of Theorem 2.4.

In Section V we shall need a result which is an immediate consequence of the above proof. This result is contained in Corollary 3.5 below. To simplify its statement we introduce the following definitions and notation. Let B_1 denote the set of all solutions, satisfying (2.1)–(2.4), of all unstable equations (1.1) for $\Delta = \Delta_c$. If $w \in B_1$ let

$$m^+(w, i) = \max_{0 \leqslant s \leqslant \Delta} w(x_i - s) \qquad \text{and} \qquad m^-(w, i) = \min_{0 \leqslant s \leqslant \Delta} w(x_i - s)$$

and define

$$\psi_i(w, \xi) = x_i - \underset{x \in [x_i - \Delta, x_i]}{\text{l.u.b}} [x : w(x) = \xi] \qquad \text{for} \qquad \xi \in [0, m^+(w, i)]$$

if i is odd and for $\xi \in [m^-(w, i), 0]$ if i is even. Finally we note that if $y, w \in B_1$ then $y_1(y)$, $w_i(w)$ will denote the zeros of y and w, respectively, as described in condition (2.1). Similar conventions will be used for the M_i and other constants associated with solutions in B_1.

COROLLARY 2.5. *Let $y \in B_1$ and z defined on $[-\Delta_c, x_j(z)]$ be such that it satisfies conditions* (2.1)–(2.4) *restricted to this interval. If $M_j(z) = M_k(y)$, $M_{j-1}(z) = M_{k-1}(y)$,*

$$\psi_j(z, \xi) \leqslant \psi_k(y, \xi) \quad \text{for} \quad \xi \in \text{ domain of } \psi_k(y, \xi),$$
$$\psi_{j-1}(z, \xi) \leqslant \psi_{k-1}(y, \xi) \quad \text{for} \quad \xi \in \text{ domain of } \psi_{k-1}(y, \xi),$$

then z may be extended to a solution in B_1 such that

$$M_{j+i}(z) = M_{k+i}(y), \; | \varphi_{j+i}(z, \xi) - \varphi_{j+i}(z, \eta)| \leqslant | \varphi_{k+i}(y, \xi) - \varphi_{k+i}(y, \eta)| \quad \text{for} \quad \xi, \eta \in I_{k+i}(y) \quad \text{and} \quad i \geqslant 0.$$

Proof. The argument is completely analogous to that given in the proof of the previous theorem.

We now define the periodic function g referred to in Section I.

$$\begin{aligned}
g(x) &= 1 - x & & x \in [0, 1\tfrac{1}{8}] \\
g(x) &= 1 - 9/8 - \int_0^{x-9/8} g(s)\, ds & & x \in [9/8, 1\tfrac{5}{8}] \\
g(x) &= -1/2 \exp[x - 1\tfrac{5}{8}] & & x \in [1\tfrac{5}{8}, 1\tfrac{5}{8} + \ln 2] \\
g(x) &= (-1)\, g(x - 1\tfrac{5}{8} - \ln 2) & & x \geqslant 1\tfrac{5}{8} + \ln 2 \\
& & & x \leqslant 0.
\end{aligned}$$

It is clear that g is a periodic function of period $3 + 1/4 + 2 \ln 2$. It is also clear that g is a solution of (1.1) for $\Delta = \Delta_c$ satisfying conditions (2.1)–(2.4) with $M_i = 1$ for $i \geqslant 1$. The function g also satisfies all but the first line of condition (2.2).

III. The Stable Equation

In this section we consider the equation (1.2) of stable type. In Theorem 3.1 we shall establish a comparison result for solutions of (1.2) when $\Delta = 3/2$.

It then follows that $\Delta = 3/2$ is the critical value of Δ for the stable equation (1.2). A result that is well known [3] for the equation (1.2c). Our proof however, allows us to describe the asymptotic behavior of all oscillatory solutions for $\Delta = 3/2$ of the stable equation (1.2).

We first define the periodic function f as follows:

$$f(x) = 1 - x \qquad x \in [0, 3/2]$$

$$f(x) = 1 - 3/2 - \int_0^{x-3/2} (1 - s)\, ds \qquad x \in [3/2, 5/2]$$

$$f(x + 5/2) = -f(x) \qquad x \geqslant 5/2, \quad x \leqslant 0.$$

We are now ready to state and prove Theorem 3.1.

THEOREM 3.1. *Let y denote any solution of the stable equation* (1.2) *for* $\Delta = 3/2$. *For any* $M > 0$ *let* $x_i > 3\Delta$ *be such that* $y(x_i) < -M/2$, $\max_{x\in[x_i, x_i-3\Delta]} | y(x)| \leqslant M$, *and for every* $\epsilon > 0$ *there exists an* $x \in [x_i - \epsilon, x_i]$ *such that* $y(x) > y(x_i)$. *Then* $y(x_i) \geqslant -M$ *and equality holds only if*

$$y(x) \equiv Mf(x - x_i + 5/2) \qquad \textit{for} \qquad x \in [x_i - 5/2, x_i].$$

Proof. If $x^* = \text{l.u.b.}\{x \mid x < x_i, y(x) = -M/2\}$ then $| x^* - x_i | < \Delta$ since for any $\epsilon > 0$ the interval $[x_i - \Delta - \epsilon, x_i]$ must contain points x at which $y(x) > 0$. If $\psi(x) = Mf(x - x^* + 3/2)$, then $\psi - y$ is monotone nonincreasing for $x \in [x^* - 3/2, x^*]$ since for such x we have

$$y'(x) \geqslant (-1) \max_{\sigma\leqslant\Delta} y(x - \sigma) \geqslant -M.$$

But since $\psi(x^*) - y(x^*) = 0$ this implies that $\psi - y$ is monotone nonincreasing in $[x^*, x^* + 1]$. Thus $-M \leqslant y(x) \leqslant 0$ for $x \in [x^* - 1/2, x_i]$. Since for every $\epsilon > 0$, $[x_i - \Delta - \epsilon, x_i]$ contains x at which $y(x) > 0$ it follows that $x_i \leqslant x^* + 1$ and so $y(x_i) \geqslant \psi(x_i)$. Now if

$$\psi(x^* - 3/2) - y(x^* - 3/2) > M\sigma > 0$$

then

$$y(x^* - 3/2 + \alpha) < M(1 - \sigma/2) \qquad \text{for} \qquad \alpha \in [0, \sigma/2]$$

and so $\psi(x^* + \sigma/2) - y(x^* + \sigma/2) < -\sigma^2 M/8$. Since $\psi - y$ is nonincreasing this implies that $y(x_i) > -M(1 - \sigma^2/8)$. Thus if $y(x_i) < -M + \epsilon$ then $| y(x) - \psi(x) | < (8\epsilon M)^{1/2}$ for $x \in [x^* - 3/2, x^*]$. If $y(x_i) < -M + \epsilon$ then since $y(x) \geqslant \psi(x)$ for x in $[x^*, x^* + 1]$ we have that $x_i \geqslant x^* + 1 - (2\epsilon/M)^{1/2}$. Thus for $x \in [x^*, x^* + 1]$ we have

$$| y(x) - \psi(x)| < \epsilon + 2M(2\epsilon/M)^{1/2} = \epsilon + (8M\epsilon)^{1/2}.$$

Collecting these results we have that if $y(x_i) < -M + \epsilon$ then for all $x \in [x^* - 3/2, x^* + 1]$

$$| \psi(x) - y(x)| \leqslant \epsilon + (8\epsilon M)^{1/2}. \tag{3.1}$$

Setting $\epsilon = 0$ completes the proof of Theorem 3.1. We note that by considering $z = -y$ we have an analogous result for the case $y(x_i) > M/2$.

COROLLARY 3.2. *For $\Delta = 3/2$ all oscillatory solutions of the stable equation* (1.2) *are bounded.*

Proof. Assume there exists an oscillatory solution y, $y(\infty) = \infty$. Select x_0, $x_0 \geqslant 3\Delta$, such that $| y(x_0)| \geqslant | y(x)|$ for all $x \in [-3/2, x_0^*]$ where x_0^* is the first $x > x_0$ for which $y(x) = 0$. But it then follows from Theorem 3.1 that $| y(x)| \leqslant | y(x_0)|$ for all $x \geqslant x_0$ which contradicts $y(\infty) = \infty$.

In order to describe the asymptotic behavior of oscillatory solutions of the stable equation (1.2) for $\Delta = 3/2$ we introduce the following definitions

(3.2) Let $\{x_i^-\}$, $\{x_i^+\}$ denote two increasing sequences of real numbers such that

$$x_0^- = -3/2, \qquad x_i^- \in [x_{i-1}^+, x_i^+] \qquad \text{for} \qquad i \geqslant 1,$$

and $\lim_{i\to\infty} x_i^+ = \infty$. Let $\{M_i\}$ be a nonincreasing sequence of positive real numbers for which $\lim_{i\to\infty} M_i \neq 0$. Let y denote any solution of the stable equation (1.1), $\Delta = 3/2$, such that for all $i \geqslant 0$, $y(x) \equiv M_i(-1)^i$ for $x \in [x_i^-, x_i^+]$ and

$$y(x) \equiv (-1)^i M_i f(x - x_i^+) \qquad \text{for} \qquad x \in [x_i^+, x_{i+1}^-].$$

The set of all such solutions corresponding to all possible choices of the sequences $\{x_i^-\}$, $\{x_i^+\}$, $\{M_i\}$, will be denoted by C.

(3.3) A solution z of (1.2) is asymptotic to a solution y of (1.2) if for every $\epsilon > 0$ there exists a $k > 0$ such that $| z(x) - y(x)| < \epsilon$ for all $x \geqslant k$.

Since $y(x) = 1$ for $x \in [-3/2, 0]$ and $y(x) = f(x)$ for $x > 0$ belongs to C we have that C is not empty.

THEOREM 3.3. *For $\Delta = 3/2$ every oscillatory solution y, $y(\infty) \neq 0$, of the stable equation* (1.2) *is asymptotic to a solution in class C.*

Proof. It follows from Corollary 3.2 that $y(\infty)$ is a finite positive number M. Since y is an oscillatory solution it follows from Theorem 3.1 that

$\lim_{x\to\infty} \sup y(x) = -\lim_{x\to\infty} \inf y(x) = M$. Thus we may select an x_0 such that $y(x_0) = M$ and $|y(x)| \leqslant M(1 + 1/8^3)$ for all $x \geqslant x_0 - 3\Delta > 0$. Let x_1 denote the first value of $x > x_0$ such that $y(x_1) = -M$ and for $i > 1$ let x_i denote the first value of $x > x_{i-1}$ for which $y(x) = (-i)^i M$. Setting $M_0 = M(1 + 1/8^3)$ and $M_i = \max_{x\in[x_i, x_{i+1}]} |y(x)|$ for $i \geqslant 1$, we define $x_i^* = \text{l.u.b}_{x\in[x_i, x_{i-1}]} [x : y(x) = M_{i-1}(-1)^i/2]$, $\quad x_{i-1}^+ = x_i^* - 3/2$, and $x_i^- \in [x_i^*, x_i^* + 1]$ such that $M_{i-1} f(x_i^- - x_i^* + 3/2) = -M_i$. Let $z \in C$ correspond to the three sequences $\{x_i^+\}$, $\{x_i^-\}$, and $\{M_i\}$ defined above. We assert that y is asymptotic to z. Since $M_i - M \geqslant 0$ and $\lim_{i\to\infty} M_i = M$ given any $\epsilon > 0$ we may select k such that $0 \leqslant M_i - M \leqslant \min[\epsilon^2/64M, \epsilon/2]$ for $i \geqslant k$. It then follows from (3.1) that $|y(x) - z(x)| \leqslant \epsilon$ for $x \in [x_i^* - 3/2, x_i^* + 1]$, $i \geqslant k + 1$. If $x_i^* + 1 < x_i^+$, then $|z(x) - y(x)|$ is monotone nondecreasing for $x \in [x_i^* + 1, x_i^+]$ and so is bounded by its value at $x_i^+ = x_{i+1}^* - 3/2$. Thus for $x \geqslant x_{k+1}^* - 3/2$ we have $|y(x) - z(x)| < \epsilon$. This completes the proof of Theorem 3.3.

In preparation for the statement and proof of Corollary 3.4 we introduce the following definitions

(3.4) A solution y of equation (1.2) will be called periodic if there exists $T > 0$ such that $y(x + T) = y(x)$ for all $x > -\Delta$. The set of all periodic solutions of the stable equation (1.2) for $\Delta = 3/2$ will be denoted by P.

(3.5) If y and z are solutions of (1.2) and if for some $\sigma > 0$, $z(x - \sigma) \equiv y(x)$ for all $x \geqslant \sigma - \Delta$, then z is called a translate of y and will be denoted by y_σ.

COROLLARY 3.4. *If $y \in P$ then for some $\sigma > 0$, $y_\sigma \in C$ and y is not a solution of* (1.1c).

Proof. Let $y \in P$ and let $z \in C$ be the solution defined in Theorem 3.3 which is asymptotic to y. It is clear from the definition of z and the fact that $y \in P$ that there exists $M > 0$ such that for $x \geqslant M$, z is periodic. Thus after an appropriate translation we may assume that $z \in P \cap C$ and z is asymptotic to y_σ for some $\sigma > 0$. Thus $y_\sigma(x) = z(x)$ for $x \geqslant -\Delta$ and $y_\sigma \in C$. Let $\{x_i^+\}$, $\{x_i^-\}$, $\{M_i\}$ be the sequences associated with y_σ. Since $M_i = M_{i+1}$ for $i \geqslant 0$ we have that $x_{i+1}^- - x_i^+ = 2\frac{1}{2}$ for $i \geqslant 0$ and $y_\sigma(x) \equiv M_0 f(x - x_i^+)(-1)^i$ for all $x \in (x_i^+, x_{i+1}^-)$. If $x_{i+1}^- = x_{i+1}^+$ then the n associated with y_σ is discontinuous at x_{i+1}^-. If $x_{i+1}^+ > x_{i+1}^-$ then $m(x) \equiv 0$ for $x \in (x_{i+1}^-, x_{i+1}^+)$ since $\min_{0\leqslant\sigma\leqslant\Delta} |y(x - \sigma)| > 0$ for any $x \in (x_{i+1}^-, x_{i+1}^+)$. Since $m(x) = 1$ for $x \in (x_i^+, x_{i+1}^-)$, m has a discontinuity at x_{i+1}. This completes the proof of Corollary 3.4.

IV. The Asymptotic Behavior of Solutions in B_3

In this section we shall consider a special subclass B_3 of solutions of the unstable equation (1.1) for $\Delta = \Delta_c$. This subclass was studied by Buchanan [2] and the results contained in Theorem 4.1 appeared in his Ph.D. Thesis. Since this result has not been published we shall, for the sake of completeness, give a proof of it here. Theorem 4.1 asserts that in the given subclass there can not exist any unbounded oscillatory solutions. In Sections V and VI we then show that if for $\Delta = \Delta_c$ the unstable equation (1.2) possesses an unbounded oscillatory solution then the class B_3 contains an unbounded oscillatory solution. Thus for $\Delta = \Delta_c$ all oscillatory solutions of the unstable equation (1.2) are bounded. Theorem 2.1 now assures that for $\Delta < \Delta_c$ all oscillatory solutions tend to zero. The results of Section II also assure us that for $\Delta < \Delta_c$ the unstable equation (1.1) possesses unbounded oscillatory solutions. It should be noted that in the subclass B_3 one may show that all oscillatory solutions y, $y(\infty) \neq 0$, are asymptotic to a scalar multiple of $g(x + \sigma)$ where g is as defined in Section II and $\sigma \in [0, 13/4 + 2 \ln 2]$. While this result is probably true for any oscillatory solution y, $y(\infty) \neq 0$, of the unstable equation (1.2) for $\Delta = \Delta_c$ we shall not consider this question. Thus we refer the reader to Buchanan's Thesis for a precise statement and proof of this fact for the class B_3.

Before defining the class B_3 we first introduce some additional definitions and notation. Let B_0 denote the set of all solutions y of the unstable equation (1.1) for $\Delta = \Delta_c$ which satisfy conditions (2.1)–(2.3), (2.4a) and the additional requirement that $| x_{i+1} - x_i | \leqslant \Delta_c$ for all $i \geqslant 0$. If $y \in B_0$ set $x_{-1} = -2$, $M_{-1} = 1$, $\delta_{-1} = [-2, 0]$. Then for $i \geqslant 0$ let

$$A_i = \min(x_{i+1} - x_i\,, \Delta_c - (x_i - x_{i-1})),$$

and let c_i denote the first point x in δ_i for which $y'(x) = y(x)$ or if no such point exists let $c_i = x_{i+1}$. Next we set $b_i = x_{i+1} - c_i$ and $a_i = x_{i+1} - x_i - b_i - A_i$. Thus $A_i(b_i\,, a_i)$ denotes the length of the interval in δ_i for which $x - n(x) = x_{i-1} (n(x) = 0, n(x) = \Delta_c)$. We now define B_3 as the set of all solutions $y \in B_0$ which satisfy the additional condition

$$A_i \geqslant \min(a_{i+1}\,, M_i/2M_{i-1}) \qquad \text{for} \qquad i \geqslant 0. \tag{4.1}$$

We are now ready to state and prove Theorem 4.1.

Theorem 4.1. *If $y \in B_3$, then $y(\infty)$ is finite.*

Proof. It follows from the definitions of A_i, b_i, and c_i that for $i \geqslant 2$

$$M_i + M_{i+1}e^{-b_i} = A_iM_{i-1} + \int_0^{a_i} |y(x_{i-1} + x)| \, dx \tag{4.2}$$

$$A_i + a_i + b_i + A_{i+1} = \Delta_c - \epsilon_i, \qquad \epsilon_i \geqslant 0. \tag{4.3}$$

Solving equation (4.2) for A_i we obtain

$$A_i = \left[M_i + e^{-b_i}M_{i+1} - \int_0^{a_i} |y(x_{i+1} + x)| \, dx\right] \Big/ M_{i-1}. \tag{4.4}$$

Summing (4.3) from 2 to n we have

$$A_{n+1} + A_2 + 2\sum_{i=3}^{n} A_i + \sum_{i=2}^{n} (b_i + a_i) = (n-1)\Delta_c - \sum_{i=2}^{n} \epsilon_i. \tag{4.5}$$

Thus we have that

$$\sum_{i=3}^{n} \Big[2M_i/M_{i-1} + (2e^{-b_i}M_{i+1}/M_{i-1} + b_i) + \left(a_i - 2\int_0^{a_i} |y(x_{i-1} + x)| \, dx/M_{i-1}\right)\Big] \leqslant (n-1)\Delta_c. \tag{4.6}$$

We first minimize with respect to b_i the expression $2e^{-b_i}M_{i+1}/M_{i-1} + b_i$. Setting its derivative equal to zero and solving for b_i one obtains $b_i = \ln[2M_{i+1}/M_{i-1}]$. Thus

$$b_i + 2e^{-b_i}M_{i+1}/M_{i-1} \geqslant \ln[2M_{i+1}/M_{i-1}] + 1. \tag{4.7}$$

Next we minimize the expression $a_i - 2\int_0^{a_i} |y(x_{i-1} + x)| \, dx/M_{i-1}$ with respect to a_i. Now if $a_i \leqslant A_{i-1}$ then

$$\int_0^{a_i} |y(x_{i-1} + x)| \, dx = M_{i-1}a_i - M_{i-2}a_i^2/2$$

and the minimum is assumed for $a_i = M_{i-1}/(2M_{i-2})$ and is $-M_{i-1}/(4M_{i-2})$. If $a_i > A_{i-1}$ then since $y \in B_3$ we have that $M_{i-2}A_{i-1} \geqslant M_{i-1}/2$ and so if $x \in [A_{i-1}, a_i]$ then $|y(x_{i-1} + x)| < M_{i-1}/2$. Thus

$$-2\int_{A_{i-1}}^{a_i} |y(x_{i-1} + x)| \, dx/M_{i-1} \geqslant -(a_i - A_{i-1})$$

and so in all possible cases we have

$$a_i - 2\int_0^{a_i} |y(x_{i-1}+x)|\,dx \geqslant -M_{i-1}/(4M_{i-2})$$

Using (4.6), (4.7) we have

$$(n-2) + \sum_{i=3}^{n} [2M_i/M_{i-1} + \ln(2M_{i+1}/M_{i-1}) - M_{i-1}/(4M_{i-2})] \leqslant (n-1)\,\Delta_c. \tag{4.8}$$

Since

$$\sum_{i=3}^{n} \ln(2M_{i+1}/M_{i-1} = (n-2)\ln 2 + \ln M_{n+1} - \ln M_2 + \ln M_n - \ln M_3$$

and

$$\sum_{i=3}^{n-1} (M_i/M_{i-1}) \geqslant (n-3)\left[\prod_{i=3}^{n-1} (M_i/M_{i-1})\right]^{1/(n-3)}$$

we obtain from (4.8) the following inequality

$$\begin{aligned}(n-2)[1+\ln 2] + (n-3)\,7[M_{n-1}/M_2]^{1/(n-3)}/4 + \ln M_{n+1}\\ \leqslant (n-1)\,\Delta_c + \ln M_2 + \ln M_3 + M_2/(4M_1).\end{aligned} \tag{4.9}$$

However if $M_{n+1} \geqslant M_2\Delta_c^2$ then $M_{n-1} \geqslant M_2$ and since $\Delta_c = 2 + 3/4 + \ln 2$ we have from (4.9) that

$$\ln M_{n+1} \leqslant \Delta_c - [1+\ln 2] + \ln(M_2M_3) + \Delta_c/4 < 2\Delta_c + \ln(M_2M_3).$$

Thus for all $n \geqslant 4$ we have that $M_n \leqslant \max(M_2\Delta_c^2, M_2M_3\exp(2\Delta_c))$. This completes the proof of Theorem 4.1.

We note that a slight modification of the above proof will suffice to establish for $\Delta = \Delta_0 < \Delta_c$ that all solution satisfying the conditions of B_3 tend to zero as $x \to \infty$. Finally we note the following corollary of Theorem 4.1.

COROLLARY 4.2. *Let y denote any solution of* (1.1) *for $\Delta = \Delta_c$ which satisfies conditions* (2.1)–(2.3) *and its restriction to the interval* $[0, x_{n+2}]$ *satisfies the conditions defining B_3. Then for any $j \leqslant n$ one has that*

$$M_j(y) \leqslant \max(M_2(y)\,\Delta_c^2, M_2(y)\,M_3(y)\exp(2\Delta_c)).$$

V. The Set B_2

In this section we shall show that if B_1 contains an unbounded solution then the set B_2 also contains an unbounded solution. This result is contained in Theorem 5.3. Here the set B_2 consists of all the solutions $y \in B_1$ of (1.1) which satisfy the following condition

$$\text{For all } x \geqslant x_1 \text{ and } i \geqslant 1 \text{ if } x \in \delta_i(y) \text{ then } x - n(x) \in \delta_i(y) \cup \delta_{i-1}(y). \tag{5.1}$$

If $z \in B_1$ then we say that $z \in B_2 \mid j$ if condition (5.1) is satisfied for all i such that $1 \leqslant i < j$. The proof of Theorem 5.3 is by induction argument. Each step in the induction corresponds to a modification of a solution in the set B_1. Thus we introduce the following definition. Let $z, y \in B_1$, then we say that z is a modification of y at (i, k) if there exists $i, k, i \leqslant k$, such that

$$M_{i+l}(z) = M_{k+l}(y) \qquad \text{for all} \qquad l \geqslant 0, \tag{5.2}$$

$$\begin{gathered} | \varphi_{i+l}(z, \xi) - \varphi_{i+l}(z, \eta)| \leqslant | \varphi_{k+l}(g, \xi) - \varphi_{k+l}(y, \zeta)| \\ \text{for any} \qquad \xi, \eta \in I_{k+l}(y) \qquad \text{and} \qquad l \geqslant 0. \end{gathered} \tag{5.3}$$

$$\max_{j \leqslant i} M_j(z) \geqslant \max_{j \leqslant k} M_j(y). \tag{5.4}$$

Before considering Theorem 5.3 we first establish the following lemmas.

Lemma 5.1. *If $y \in B_1$, $y(\infty) = \infty$, then there exists $z \in B_1$, z a modification of y at $(3, 3 + 2k)$ where $k \geqslant 0$, and $z \in B_2 \mid 3$.*

Proof. Since $y \in B_1$ we have since $y(x) \leqslant 1$ for $x \leqslant 0$ that $y \in B_2 \mid 2$. If $y \in B_2 \mid 3$ we are finished, thus we shall assume that $y \notin B_2 \mid 3$. We consider first the case in which $M_2(y) < M_0$. Now if $m^+(y, 2) = M_2 < M_0$ or $m^-(y, 1) = -M_1$ then $y \in B_2 \mid 3$ which is a contradition. Let j denote the first even integer greater than 2 for which $M_j > M_2$ or $M_{j-1} > M_1$. If $M_{j-1} > M_1$ then we define $z \in B_1$ such that $z(x) = y(x)$ for $x \leqslant x_1(y)$ and $M_{1+i}(z) = M_{j-1+i}$ for $i \geqslant 0$. Since j was the first even integer for which $M_{j-1} > M_1$, we have, since $m^-(y, 1) < -M_1$ and $y(x) < M_0$ for $x \in (x_0(y), x_{j-1}(y))$, that $\psi_1(z, \xi) \leqslant \psi_{j-1}(y, \xi)$ for all $\xi \in [0, m^+(y, j-1)]$. Since $x_1(z) < x_{j-1}(y)$ and $y(x) > -M_{j-1}(y)$ for $x \in [0, x_{j-1}(y))$, it then follows that $\psi_2(z, \xi) \leqslant \psi_j(y, \xi)$ for all $\xi \in [m^-(y, j), 0]$. By Corollary 2.5 it follows that z is a modification of y at $(1, j-1), j > 2$.

Next we consider the case in which $M_j < M_2$. In this case we set $M_{2+i}(z) = M_{j+i}(y)$ for $i \geqslant 1$. Since $m^+(y, 2) > M_2(y)$ and $M_{j-2i-1} \leqslant M_1$ for $0 \leqslant 2i < j - 1$, we have that $\psi_2(z, \xi) \leqslant \psi_2(y, \xi)$ for $\xi \in [m^-(y, j), 0]$. Since $M_{j-2i} \leqslant M_2$, $0 \leqslant 2i < j$, we then have that $\psi_3(z, \xi) \leqslant \psi_{j+1}(y, \xi)$ for $\xi \in [0, m^+(y, j+1)]$. By Corollary 2.5 it follows that z is a modification of y at

$(2, j)$. Since y is unbounded and $M_k \leqslant [\max_{2j \leqslant k-1} M_{k-(1+2j)}] \cdot \Delta_c$ it is clear that after a finite number of steps we must have $M_2 > M_0$ or $z \in B_2 \mid 3$.

Thus we consider the case in which $M_2 \geqslant M_0$. Let $M^* > M_1(y)$ denote $| m^-(y, 2)|$ and let $z(x) = y(x)$ for $x \leqslant x_1(y)$, $M_1(z) = M^*$, $M_i(z) = M_i(y)$ for $i \geqslant 2$. Since $M^* \leqslant M_0$ and $|y(x)| \leqslant M_0$ for $x \leqslant 0$ it follows that $z'(x) \geqslant M_0$ for $x \in [x_1(z), x_2(z)]$ and z is well defined for $x \leqslant x_2(z)$. Thus $\psi_2(z, \xi) \leqslant \psi_2(y, \xi)$ for $\xi \in [-M^*, 0]$ and since $M^* = m^-(z, 2) = -M_1(z)$ we have that $| \varphi_2(z, \xi) - \varphi_2(z, \eta)| \leqslant | \varphi_2(y, \xi) - \varphi_2(y, \eta)|$ for $\xi, \eta \in I_2(y)$. It then follows from Corollary 2.5 that z is a modification of y at $(2, z)$ and clearly $z \in B_2 \mid 3$. This completes the proof of Lemma 5.1.

LEMMA 5.2. *If $y \in B_2 \mid p$, $p \geqslant 3$ and $y(\infty) = \infty$, then there exists a modification z of y at $(p, p + m)$, $m \geqslant 2$, and $z \in B_2 \mid p$* or *at (p, p) and $z \in B_2 \mid p + 2$.*

Proof. We may assume without loss of generality that p is odd and when no confusion is possible we write M_i and x_i for $M_i(y)$ and $x_i(y)$. If $M_{p-1} \geqslant M_{p-3}$ then since $y \in B_2 \mid p$ we would have that $m^+(y, p - 1) = M_{p-1}$ and so $y \in B_2 \mid p + 1$. Thus we shall assume that $M_{p-1} < M_{p-3}$. We next note that we may assume that $M_p \geqslant M_{p-2}$. For we assume $M_p < M_{p-2}$ and let m denote the first even integer such that $M_{p+m} \geqslant M_p$ or $M_{p+m-1} \geqslant M_{p-1}$. Since $y(\infty) = \infty$ such an integer exists. In the first case we define $z(x) = y(x)$ for $x \leqslant x_p(y)$ and $M_{p+i}(z) = M_{p+m+i}$ for all $i \geqslant 0$. In the second case we define $z(x) = y(x)$, $x \leqslant x_{p-1}$, and $M_{p-1+i}(z) = M_{p+m+i-1}$ for all $i \geqslant 0$. The argument used to show that z is a modification of y at $(p, p + m)$ in the first case and at $(p - 1, p + m - 1)$ in the second case is completely analogous to that given in the proof of Lemma 5.1 and will not be repeated. Since $y \in B_2 \mid p$ it is clear in both cases that $z \in B_2 \mid p$.

Thus we are reduced to the case in which $M_{p-1} < M_{p-3}$ and $M_p > M_{p-2}$. Since $y \in B_2 \mid p$ and $y \notin B_2 \mid p + 1$ we have that $m^+(y, p) \in [M_{p-3}, M_{p-1})$. Then if

$$[m^+(y, p) + M_{p-2}]/|x_{p-2} - \varphi_{p-3}(y, m^+(y, p))] \leqslant M_{p-2} \tag{5.5}$$

we define $z(x) = y(x)$ for $x \leqslant x_{p-1}$, $M_{p-1}(z) = m^+(y, p)$, $M_{p+i}(z) = M_{p+i}(y)$ for $i \geqslant 0$. Since $M_p > M_{p-2}$ and $y \in B_2 \mid p$ it is clear that $x_p(y) - x_{p-1}(y) > 1$ and since $m^+(y, p) = M_{p-1}(z)$ we also have that $x_{p-1}(z) - x_{p-1}(y) < 1$. From (5.5) we have

$$\begin{aligned}[m^+(y, p) + M_{p-2}]/M_{p-2} + [x_{p-1}(y) - x_{p-2}(y)] + [M_{p-1} + M_{p-2}]/M_{p-2} \\ + \ln(M_p/M_{p-2}) \leqslant \Delta_c\end{aligned} \tag{5.6}$$

from which it follows, since $x_{p-1}(z) - x_{p-1}(y) \leqslant 1$, that

$$x_{p-1}(z) - x_{p-2}(z) + [m^+(y, p) + M_{p-2}]/M_{p-2} + \ln M_p/M_{p-2} < \Delta_c .$$

This last statement implies that $x_p(z) - x_{p-2}(z) < \Delta_c$ from which it follows that $\psi_p(z, \xi) \leqslant \psi_p(y, \xi)$ for $\xi \in [0, m^+(y, p)]$. Since $M_p > M_{p-2}$ and $y \in B_2 \mid p$ this in turn implies that $\psi_{p+1}(z, \xi) \leqslant \psi_{p+1}(y, \xi)$ for $\xi \in [m^-(y, p+1), 0]$. Thus by Corollary 2.5 z is a modification of y at (p, p). It is clear since $y \in B_2 \mid p$ and $M_{p-1}(z) = m^+(z, p)$ that $z \in B_2 \mid p + 2$.

Thus we may assume that

$$[m^+(y, p) + M_{p-2}]/[x_{p-2} - \varphi_{p-3}(y, m^+(y, p))] > M_{p-2}. \tag{5.7}$$

We shall consider first the cases in which $M_{p+1} \geqslant M_{p-1}$. We introduce the following notation (Figure 1):

$$k = M_{p-1}/M_{p-2}, \quad h = M_{p-3}/M_{p-2}, \quad M = m^+(y, p) = (1 + g)\, M_{p-1},$$
$$a = \varphi_{p-3}(y, M) - x_{p-3}(y), \qquad b = x_{p-2}(y) - x_{p-3}(y) - a,$$
$$c = x_{p-1}(y) - x_{p-2}(y), \qquad d = x_p(y) - x_{p-1}(y).$$

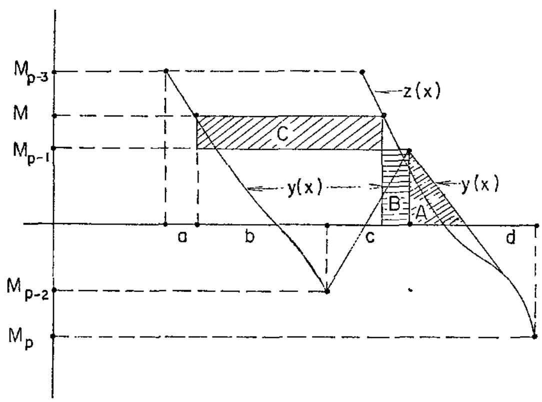

FIG. 1

Now if $k > 1$ then $d > 2$ and so $b < 3/4 + \ln 2 < 1 + 1/2$. In this case set $z(x) = y(x)$ for $x \leqslant x_{p-3}$ and $M_{p-2+i}(z) = M_{p+i}(y)$ for $i \geqslant 0$. Translating the curve $z(x)$ so that $x_{p-2}(z) = x_p(y)$ (confer Figure 2) denote their point of intersection by q. Since $b < 1 + 1/2$ and $x_p(y) - y_p = 1$ it is clear that $y_p - q < 1/2$. Then the area C bounded by the curves $y(x)$, $z(x)$ and the x-axis is less than the area E bounded by the curves $y(x)$, $z(x)$ and the line $y = M_{p-1}$ since $y_p - x_{p-1}(y) > 1$. It then follows that $| \varphi_p(y, \xi) - \varphi_p(y, \eta)| \geqslant | \varphi_{p-2}(z, \xi) - \varphi_{p-2}(z, \eta)|$ for $\xi, \eta \in I_p(y)$. Since $M_p > M_{p-2}$ and $y \in B_2 \mid x_p(y)$ it is clear for $x \in \delta_{p+1}(y)$ that

$$x - n(x) \in \delta_{p+1}(y) \cup \delta_p(y).$$

This fact along with the preceeding inequality gives that

$$| \varphi_{p+1}(y, \xi) - \varphi_{p+1}(y, \eta)| \geqslant | \varphi_{p-1}(z, \xi) - \varphi_{p-1}(z, \eta)|$$

for all $\xi, \eta \in I_{p+1}(y)$. Since $M_{p+1} \geqslant M_{p-1}$ it then follows by Corollary 2.5 that z is a modification of y at $(p-2, p)$ and $z \in B_2 \mid p$. We note that if $x_{p-2} - y_{p-2} \leqslant 1$ then the point of intersection q in Figure 2 lies to the right of the intersection of the curves with the x axis. In this case c is zero and we

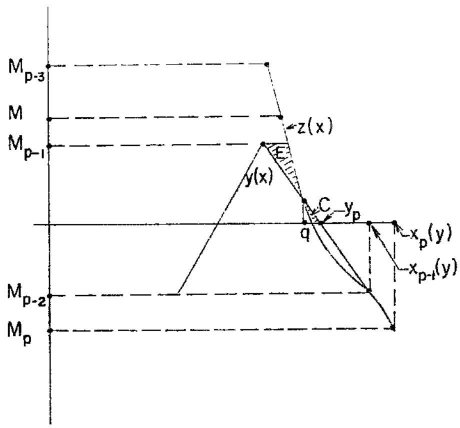

FIG. 2

again obtain a solution z which is a modification of y at $(p-2, p)$ and $z \in B_2 \mid p$. In fact in this case the restriction $M_{p+1} \geqslant M_{p-1}$ is not necessary. Thus we shall assume that $k \leqslant 1$ and $b \geqslant 1$. We next note that if $g < 1/2$ then we set $z(x) = y(x)$ for $x \leqslant x_{p-1}(y)$, $M_{p-1}(z) = M$, and $M_{p+i}(z) = M_{p+i}(y)$ for $i \geqslant 0$. Since $b \geqslant 1$ we have that $x_{p-1}(z) - x_{p-1}(y) < 1/2$ and since $k \leqslant 1$ we have that $x_p(z) - x_{p-1}(z) \leqslant x_p(y) - x_{p-1}(y) + 1/2$. Since $b \geqslant 1$ we then have that $y(x_p(y) - s) = z(x_p(z) - s)$, for $0 \leqslant s \leqslant x_p(y) - x_{p-1}(y)$. Thus $\psi_p(y, \xi) \geqslant \psi_p(z, \xi)$ for all $\xi \in [m^+(y, p), 0]$ and $| \varphi_p(z, \xi) - \varphi_p(z, \eta)| \leqslant | \varphi_p(y, \xi) - \varphi_p(y, \eta)|$ for all $\xi, \eta \in I_p(y)$. Since $M_p > M_{p-2}$ we also have $\psi_{p+1}(y, \xi) \geqslant \psi_{p+1}(z, \xi)$ for all $\xi \in [m^-(y, p+1), 0]$. It now follows from Corollary 2.5 that z is a modification of y at (p, p). It is clear that $z \in B_2 \mid p + 2$.

We now show that if $k \leqslant 1$, $g \leqslant 1/2$, $b > x_{p-2} - y_{p-2} > 1$ then we may define $z(x) = y(x)$ for $x \leqslant x_{p-2}$, $M_{p-2}(z) = M_p(y)$ and $M_{p+i-2}(z) = M_{p+i}(y)$ for $i \geqslant 1$. Referring to Figure 1 and the discussion in the case $k > 1$ it will suffice to show that $C > B + A$ where C and B are the areas of the indicated rectangles and A the area of the indicated triangle. But $C = gkM_{p-2}(c + d)$ and $A + B \leqslant (k^2M_{p-2})/2 + [b - (1 + k)]\, kM_{p-2}$. Furthermore

$$[b - (1 + k)] \leqslant [M - M_{p-1}]/M_{p-2} = (1 + g)\, k - k = gk$$

and so $A + B \leqslant k^2M_{p-2}/2 + gk^2M_{p-2}$. For c and d we have the estimates $c \geqslant M_{p-2}(1 + k)/(hM_{p-2}) = (1 + k)/h$ and $d = 1 + k$. Thus we have $C - [A + B] \geqslant kM_{p-2}[g[1 + (k + 1)/h] - k/2] > 0$ since $k \leqslant 1$ and $g \geqslant 1/2$. It then follows as before that z is a modification of y at $(p - 2, p)$ and $z \in B_2 \mid p$.

In the preceeding discussion the restriction $M_{p+1} \geqslant M_{p-1}$ was necessary to assure us that for $x \geqslant x_{p+1}$, $x - n(x) \notin [x_{p-1}, y_p]$. Thus if $m^+(y, p + 2) \leqslant M_{p+1}$ the discussion for the case $M_{p+1} \geqslant M_{p-1}$ is valid for the case $M_{p+1} < M_{p-1}$. Thus we may assume that $m^+(y, p + 2) > M_{p+1}$. We also note that we may assume that $M_{p+2} > M_p$. For if this is not the case we select the first even integer such that $M_{p+2+k} > M_{p+2}$ or $M_{p+1+k} > M_{p+1}$ and obtain a solution z which is a modification of y at $(p + 2, p + 2 + k)$ in the first case and $(p + 1, p + 1 + k)$ in the second. The argument is the same as in the case $M_p < M_{p-2}$, $M_{p-1} < M_{p-3}$ and will not be repeated. Since $y(\infty) = \infty$, we must after a finite number of steps have that either $M_{p+1} > M_{p-1}$ or $M_{p+2} < M_p$. Since the case $M_{p+1} > M_{p-1}$ has been consider earlier we shall assume $M_{p+2} > M_p$. Let $M^* = m^+(y, p + 2) > M_{p+1}$. Then define $z(x) = y(x)$ for $x \leqslant x_{p+1}$, $M_{p+1}(z) = M^*$, $M_{p+i}(z) = M_{p+i}$ for $i \geqslant 2$. Since $y \in B_2 \mid p$ and $M_{p+2} > M_p > M_{p-2}$ we have that $x_p - y_p > 1$ and $x_{p+2} - y_{p+2} > 1$. Thus $z'(x) \geqslant M^*$ for $x \in [x_{p+1}(y), x_{p+1}(z)]$ and so $x_{p+1}(z) - x_{p+1}(y) < 1$. Since

$$\Delta \geqslant M^*/M_{p-2} + 1 + x_{p+1} - x_p + M_{p+1}/M_p + 1 + \ln(M_{p+2}/M_p)$$

we have that

$$\begin{aligned} &x_{p+2}(z) - x_p(z) \\ &\qquad \leqslant x_{p+1} - x_p + 1 + M^*/M_p + M_{p+1}/M_p + 1 + \ln(M_{p+2}/M_p) < \Delta. \end{aligned}$$

Thus $m^+(y, p + 2) = m^+(z, p + 2) = M_{p+1}(z)$, $m^-(y, p + 2) = M_{p+2} = M_{p+2}(z) = m^-(z, p + 2)$ and $\psi_{p+2}(z, \xi) \leqslant \psi_{p+2}(y, \xi)$ for $\xi \in [0, M^*]$. It then follows that z is a modification of y at $[p + 2, p + 2]$. But now $m^+(z, p + 2) = M_{p+1}(z)$ and we are reduced to the discussion for the case $M_{p+1} > M_{p-1}$. But if z is a modification of y at $(p - i, p + m - i)$, $i = 1, 2$, then it is also at $(p, p + m)$. This completes the proof of Lemma 5.2.

THEOREM 5.3. *If there exists* $y \in B_1$, $y(\infty) = \infty$, *then there exists a* $z \in B_2$, $z(\infty) = \infty$.

Proof. Let z_0 denote the modification of y at $(3, 3 + 2k)$ such that $z \in B_2 \mid 3$. If $z_i \in B_2 \mid p(i)$, $i \geqslant 0$ and $p(i)$ maximal, let z_{i+1} denote the modification of z_i as given in Lemma 5.2. Since $y(\infty) = \infty$, for an infinite number of i we have $p(i + 1) \geqslant p(i) + 2$. Thus $p(i)$ is a monotone non-decreasing function of i and $\lim_{i\to\infty} p(i) = \infty$. Since $y(\infty) = \infty$ the max $M_j(z_i)$ for $j \leqslant p(i)$ tends to ∞ with i and we have that $\lim_{i\to\infty} x_{p(i)}(z_i) = \infty$. We note however in the proof of Lemma 5.2, that $z(x) = y(x)$ for $x \leqslant x_{p-2}(y)$. Thus the z_i converge on $[-\Delta, \infty)$ to a solution $z \in B_2$, $z(\infty) = \infty$. This completes the proof of our theorem.

VI. The Set B_3

In this section we shall show in Theorem 6.3 that if there exists a solution $y \in B_2$, $y(\infty) = \infty$, then there exists for every $M > 0$ a solution $w \in B_0$ such that for some $j > 0$ one has $M_{j+1}(w) > M$ and $w \in B_3 \mid j$. If $w \in B_0$ we say that $y \in B_3 \mid j$ if condition 4.1 is satisfied for $1 \leqslant i \leqslant j$. For M sufficiently large, the existence of such a solution $w \in B_0$ gives a contradiction of Corollary 4.2. Thus we show in Theorem 6.5 that if $\Delta = \Delta_c$ then every oscillatory solution of the unstable equation (1.2) is bounded. Finally we note in Theorem 6.6 that if there exists a solution $y \in B_2$, $y(\infty) = \infty$, then there exists a solution $w \in B_3$, $w(\infty) = \infty$. The construction of such a w is similar to the construction given in Theorem 2.3 using of course the results of Theorem 6.3.

We introduce the following definition. If $y, z \in B_0$ we shall say that z is a modification of y at $[i, j]$, $i \leqslant j$, if:

$$M_{i+k}(z) = M_{j+k}(y) \qquad \text{for} \qquad k \geqslant 1, \tag{6.1}$$

$$z(x) \equiv y(x) \qquad \text{for} \qquad x \in [-\Delta_c, 0] \tag{6.2}$$

$$| \varphi_{i+k}(z, \xi) - \varphi_{i+k}(z, \eta)| \leqslant | \varphi_{j+k}(y, \xi) - \varphi_{j+k}(y, \xi)| \quad \text{for} \quad \xi, \eta \in I_{j+k}(y),\quad k \geqslant 2. \tag{6.3}$$

We now note the following modification of Corollary 2.5 for solutions in B_0.

Corollary 2.5a. *Let $y \in B_0$ and z defined on $[-\Delta_c, x_{i+2}(z)]$ be such that it satisfies the conditions defining B_0 restricted to this interval and $M_{i+1}(z) = M_{j+1}(y)$, $M_{i+2}(z) = M_{j+2}(y)$, If $i \leqslant j$ and $\psi_{i+2}(z, \xi) \leqslant \psi_{j+2}(y, \xi)$ for all ξ in the domain of $\psi_{j+2}(y)$, then z may be extended to be a modification of y at $[i, j]$.*

To simplify the proof of Theorem 6.3 we shall first prove the following two lemmas. For $i \geqslant 1$ let $s_i(y) = \lim_{\epsilon\to 0+} y(x_i - \epsilon)$ and $u_i(y) = \min_{x\in\delta_i} | y'(x)|$.

Throughout this section, except where confusion might arise, we shall write s_i, u_i, M_i,... for $s_i(y)$, $u_i(y)$, $M_i(y)$,.... We now have the following results.

LEMMA 6.1. *If* $y \in B_0$ *and*
$I(i+1) = [(1 + M_i/s_{i+1})(1/u_{i+1} - 1/M_i) \min(A_{i+1}M_i, M_{i+1} - u_{i+2})] > A_{i+2}$ *then there exists* $\sigma > 0$ *and* $z \in B_0$ *such that* $M_{i+1}(z) = M_{i+1} - \sigma$, *and* z *is a modification of* y *at* $[i+1, i+1]$. *If for any* $j \geqslant 0$ y *satisfies Condition* (4.1) *for* $i = j$ *then* z *also satisfies Condition* (4.1) *for* $i = j$.

Proof. We shall assume that i is odd, the argument for i even is the same. For $x \leqslant \varphi_i(y, M_{i+1} - \sigma)$ we define $z(x) = y(x)$. Setting $M_{i+1}(z) = M_{i+1} - \sigma$, $M_{i+k}(z) = M_{i+k}$ $(k = 2, 3)$ we extend the definition of z to $x = x_{i+3}(z)$. Since $x_{i+2}(z) - x_{i+1}(z) < x_{i+2}(y) - x_{i+1}(y)$ it will follow from Corollary 2.5a that z is a modification of y at $[i+1, i+1]$ if we can show that

$$\psi_{i+3}(z, \xi) \leqslant \psi_{i+3}(y, \xi) \qquad \text{for} \qquad \xi \in [-M_{i+2}, 0]. \tag{6.4}$$

After an appropriate translation we may assume that $x_{i+2}(z) = x_{i+2}(y)$. We first note for any $0 \leqslant \sigma < M_{i+1}$ that $z(x) \geqslant y(x)$ for $x \in \delta_{i+1}(z)$. This is an immediate consequence of the definition of z and the fact that

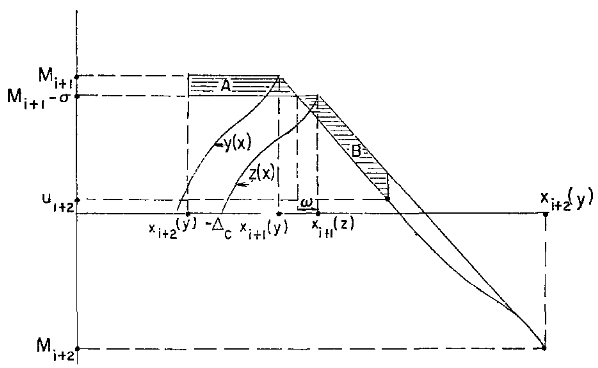

FIG. 3

$M_{i+1}(z) = M_{i+1} - \sigma$. Referring to Figure 3, the desired inequality (6.4) will follow if we can show that the area B exceeds the area A. However $A = [A_{i+1} + \sigma/(M_i 2)]\, \sigma$ and $B = bw - w^2/2M_i$ where

$$b = \min(A_{i+1}M_i, M_{i+1} - u_{i+2})$$

and

$$w = \psi_{i+2}(z, M_i - \sigma) - \psi_{i+2}(y, M_i - \sigma).$$

If we can show that $w = \sigma(1 + M_i/s_{i+1})(1/u_{i+1} - 1/M_i) + O(\sigma^2)$ it will then follow from the hypothesis of our lemma that for σ sufficiently small, $B > A$. To obtain the desired estimate for w first note that by decreasing M_{i+1} by σ one increases A_{i+1} by $\sigma/s_{i+1} + O(\sigma^2)$ and so traverses a segment of length $M_i[(\sigma/s_{i+1}) + O(\sigma^2)]$ in $[M_{i+1}(z), -M_{i+2}(z)]$ at the rate M_i instead of $u_{i+1} + O(\sigma)$. Thus the decrease in $\psi_{i+2}(z, M_{i+1} - \sigma)$ as compared to $\psi_{i+2}(y, M_{i+1} - \sigma)$ is

$$M_i\sigma(1/u_{i+1} - 1/M_i)/s_{i+1} + O(\sigma^2).$$

In addition we have that $M_{i+1}(z) = M_{i+1} - \sigma$ so that z covers a segment of length σ in $[M_{i+1} - \sigma, -M_{i+2}]$ at the rate M_i as compared to the rate $u_{i+1} + O(\sigma)$ for y. The resulting decrease in $\psi_{i+2}(z, M_{i+1} - \sigma)$ is $\sigma(1/u_{i+1} - 1/M_i) + O(\sigma^2)$. Thus for σ sufficiently small $B > A$ and z is a modification of y at $[i + 1, i + 1]$. The second assertion of the lemma then follows for $j \geqslant i + 3$. But since $A_{i+1}(z) > A_{i+1}$, $a_{i+2}(z) < a_{i+2}$, $a_{i+3}(z) < a_{i+3}$, $A_{i+2}(z)\, M_{i+1}(z) > A_{i+2}M_{i+1}$ the result follows for $j = i + 1, j = i + 2$. For $j \leqslant i$ it follows from the fact that $z(x) = y(x)$ for $x \leqslant x_{j+1}(z)$ and $a_{j+1}(y) > a_{j+1}(z)$. This completes the proof of our lemma.

Lemma 6.2. *If*

$$y \in B_0, \quad u_{i+3} \geqslant 1/2, \quad M_{i+2} = 1 \quad \textit{and} \quad A_{i+3} > u_{i+3}(1 + A_{i+2})/u_{i+2}$$

for some $i \geqslant 0$, then there exists $\sigma > 0$ and $z \in B_0$ such that $M_{i+1}(z) = M_{i+1} - \sigma$, $M_{i+2}(z) = M_{i+2} + \sigma$ and z is a modification of y at $[i + 2, i + 2]$. If y satisfies Condition 4.1 for any integer $j \geqslant 0, j \neq i + 1, i + 2, i + 3$, then z will also satisfy Condition 4.1 for the integer j.

Proof. We may without loss of generality assume that i is even. For any small positive σ we define $z(x) = y(x)$ for $x \leqslant \varphi_i(y, -M_{i+1} + \sigma)$, $M_{i+2}(z) = M_{i+2} + \sigma$, $M_{i+1}(z) = M_{i+1}(y) - \sigma$, $M_j(z) = M_j$ for $j \geqslant i + 3$. We first note that $| x_{i+1}(z) - x_i(z)| \leqslant | x_{i+1} - x_i |$ and so $A_{i+2}(z) \geqslant A_{i+2}$. Since z is a modification of y at $[i + 2, i + 2]$ the last statement of our lemma will then follow for $j \geqslant i + 4$. For $j < i + 1$ the result follows from the definition of z along with the fact that $a_{i+1}(z) \leqslant a_{i+1}$. Thus we must establish that for σ sufficiently small z is a modification of y at $[i + 2, i + 2]$. Since $| x_{i+2}(z) - x_{i+1}(z)| \leqslant | x_{i+2} - x_{i+1} |$ we have that

$$\begin{aligned}\varphi_{i+2}(z, -(u_{i+2} - A_{i+2}\sigma - \sigma)) - x_{i+2}(z) \\ = \varphi_{i+2}(y, -u_{i+2}) - x_{i+2} + O(\sigma^2)\end{aligned}$$

and so

$$\begin{aligned}\varphi_{i+2}(z, -u_{i+2}) - x_{i+2}(z) \\ = \varphi_{i+2}(y, -u_{i+2}) - x_{i+2} + \sigma(1 + A_{i+2})/u_{i+2} + O(\sigma^2).\end{aligned}$$

Thus $A_{i+3} - A_{i+3}(z) = \sigma(1 + A_{i+2})/u_{i+2} + O(\sigma^2)$.

Consulting figures 4 and 5 (where we have set $x_{i+3}(z) = x_{i+3}$) we see that if the area A exceeds the area C and also exceeds the area B then

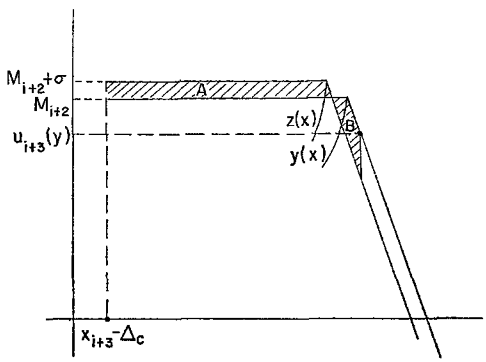

FIG. 4

$u_{i+3}(z) > u_{i+3}(y)$, and $\psi_{i+4}(z, \xi) \leqslant \psi_{i+4}(y, \xi)$ for ξ in the domain of $\psi_{i+4}(y)$. Here $A = A_{i+3}\sigma + O(\sigma^2)$, $C = u_{i+3}[(1 + A_{i+2})/u_{i+2}]\,\sigma + O(\sigma^2)$ and $B = [(1 - u_{i+3})(1 + A_{i+2}/u_{i+2})]\,\sigma + O(\sigma^2) \leqslant [u_{i+3}(1 + A_{i+2})/u_{i+2}]\,\sigma + O(\sigma^2)$. Thus for σ sufficiently small we have by Corollary 2.5a that z is a modification of y at $[i + 2, i + 2]$. This completes the proof of Lemma 6.2.

We are now ready to state and prove Theorem 6.3.

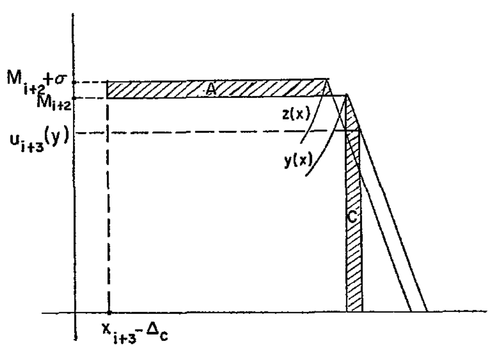

FIG. 5

THEOREM 6.3. *If there exists* $y \in B_2$, $y(\infty) = \infty$, *then for any* $M > 1$ *there exists* $z \in B_0$, $z \in B_3 \mid i - 1$, $M_i(z) \geqslant M$.

Proof. If $j = 0, 1, 2,\ldots$ we denote by $B_j(M, i)$ the set of all solutions $z \in B_j$ for which $M_k(z) \geqslant M$ for some $k \leqslant i$. Let $i(M, B_j)$ denote the smallest positive integer for which $B_j(M, i)$ is not empty. It follows from the hypothesis of our theorem that $B_2(M, i)$ is not empty for i sufficiently large. Let $y \in B_2(M, i)$ then if $y \notin B_0(M, i)$ we may construct a $z \in B_0(M, i)$. For let $k \leqslant i$ be the first integer for which $M_k(y) \geqslant M$. We then construct the solution $z \in B_0$ having the following $M_i(z)$. We set $M_k(z) = M_k(y)$ and for $i < k$ we set $M_i(z) = m^+(y, i + 1)$ if i is even and $M_i(z) = m^-(y, i + 1)$. if i is odd. Assuming that the $M_i(z)$ have been defined for $i \leqslant l$, $l \geqslant k$, we define $M_{l+1}(z)$ so that $M_l A_{l+1}(z) = 3/2 M_{l+1}(z)$. Since $z \in B_0(M, i)$ we have that $i(M, B_0) \leqslant i(M, B_2)$. One could in fact show that $i(M, B_j) = i(M, B_{j+1})$ for $j = 0, 1$. However for our purposes it suffices to know that $B_0(M, i)$ is non-empty for i sufficiently large. Thus if $j = i(M, B_0)$ we now prove by induction that given any $y \in B_0(M, j)$, for every $k \leqslant j - 1$, there exists $z \in B_0(M, j)$ such that z is a modification of y at $[k, k]$ and $z \in B_3 \mid k$. Let $y \in B_0(M, j)$ then since $A_0 > 1$ we have that $y \in B_3 \mid 0$. Since $A_0 > 1$ it also follows that $\int_0^{a_1} y(x + \sigma)\, d\sigma < 1/2$ so if $M_1 \geqslant 1$ we must have $A_1 M_0 > M_1/2$. But if $M_1 \leqslant 1$ then $A_1 > 9/8$ and so in every case $y \in B_3 \mid 1$.

We now consider our induction step. We assume that for every $y \in B_0(M, j)$ and $k < l \leqslant j - 1$ there exists a $z \in B_0(m, j)$ such that z is a modification of y at $[k, k]$, $z \in B_3 \mid k$, and if for any $j > k$ y satisfies condition (4.1) then z also satisfies condition (4.1) for this value of j. We then establish this result for $k = l$. The proof proceeds as follows. Let $y \in B_0(M, j)$ then by our induction hypothesis there is a $z \in B_3 \mid l - 1$ which is a modification of y at $[l - 1, l - 1]$. Now using Lemmas 6.1 and 6.2 we obtain a modification w of z at $[l, l]$. This w will satisfy condition (4.1) for $i = l$ and for all other values of i at which z does with the possible exception of $l - 3$, $l - 2$. Then applying our induction hypothesis we obtain a w_1 which is a modification of y at $[l, l]$ and $w_1 \in B_3 \mid l$. The exceptions $l - 3$, $l - 2$ result from an application of Lemma 6.2 in cases IId, IIe described below. Thus we shall also show that these cases cannot arise for $l \leqslant 3$. We now proceed with our proof. For notational convenience we shall take $l = 4$. We note that the discussion at the beginning of IIc assures us that if $M_3 \geqslant (3/2) M_2$ then $A_3 M_2 \geqslant M_3$. Thus it follows that after applying Lemma 2.2 in IIe and IId the modified solution z still satisfies condition (4.1) for $i = 3$ (written $z \in (4.1, 3)$). This is why $l - 1$ was not one of the exceptions noted above. Thus we assume that we have a solution $y \in B_3 \mid 3$ and show that by repeated application of Lemmas 6.1 and 6.2 we may obtain a solution z which is a modification of y at $[4, 4]$ and satisfies condition (4.1) for $i = 4$. In order to apply these lemmas the necessary inequatities must be obtained. It should be noted that inobtaining these inequalities the independent variables (usually M_i, M_{i+1}) will be restricted to closed domains and both sides of the inequalities considered will be continuous functions of these variables. Thus for example in all applications of Lemma 6.1, u_{i+1} and M_i will be bounded away from zero. This means that for a given y the σ in Lemma 6.1 and 6.2 could be chosen uniformly and our modified solution z is thus obtained after a finite number of applications of Lemma 6.1 and 6.2.

For the sake of clarity we shall divide the discussion into the following cases.

I. $M_2 \geqslant M_4$;

IIa. $M_2 \leqslant M_4$, $M_3 \leqslant M_2$;

IIb. $M_2 \leqslant M_4$, $M_3 \geqslant M_2$, $u_3 \leqslant M_2/2$;

IIc. $M_2 \leqslant M_4$, $M_2 \leqslant M_3 \leqslant 3M_2/2$, $u_3 \geqslant M_2/2$;

IId. $M_2 \leqslant M_4$, $M_3 \geqslant 3M_2/2$ and $u_3 \geqslant 2M_2/3$ or $u_3 \geqslant M_2/2$ and $u_2 \geqslant 3/2M_2$;

IIe. $M_3 \geqslant 3/2M_2$, $M_2/2 \leqslant u_3 \leqslant 2M_2/3$, $u_2 \leqslant 3/2M_2$, $M_2 \leqslant M_4$.

Before discussing each of the above cases we first prove the following lemma.

Lemma 6.4. *If* $y \in B_0(M, j)$, $j = i(M, B_0)$ *then for* $i + 1 < j$ *either* $y(x_i + A_i + a_i) = y(x_{i-1} + a_i)$ *or* $M_{i+1} = |y(x_i + A_i + a_i)| > 3M_i/7$.

Proof. Assume $M_{i+1} \leqslant 3M_i/7$ and $|y(x_i + A_i + a_i)| < |y(x_{i-1} + a_i)|$ Thus $|y'(x)| > M_{i+1}$ for $x \in [x_i, x_{i+1}]$. Let j denote the first even integer for which $M_{i+j} > M_i$ or $M_{i+j+1} > M_{i+1}$. If $j = 2$ and $M_{i+2} < M_i$ or if $j > 2$ we may obtain a solution z which is a modification of y at $[i + 2, i + j]$ or $[i + 1, i + j + 1]$ which would contradict the fact that $j = i(M, B_0)$. Thus we may assume that $M_{i+2} \geqslant M_i$. We now show that the condition $M_{i+1} \leqslant 3M_i/7$ gives a contradiction. If we set $M_{i+1} = \gamma M_i$, $\gamma \leqslant 3/7$, we have that $x_{i+2} - x_{i+1} \geqslant 1 + \gamma$ and so $A_{i+2} \leqslant \Delta_c - (1 + \gamma)$. Thus we have that

$$\begin{aligned} M_{i+2} &\leqslant A_{i+2}M_{i+1} + \int_0^{a_{i+2}} |y(x_{i+1} + \sigma)|\, d\sigma \leqslant M_{i+1}[\Delta_c - (1 + \gamma)] \\ &\quad + \int_0^{\gamma} (M_{i+1} - \sigma M_i)\, d\sigma \leqslant M_{i+1}[\Delta_c - 1] - \gamma^2/2M_i \\ &\leqslant [2 + 1/2\gamma - \gamma^2/2]\, M_i . \end{aligned}$$

However, for $\gamma \leqslant 3/7$ the quantity in brackets is less than 1 and we obtain a contradiction. This completes the proof of Lemma 6.4 and we return to our proof of Theorem 6.3.

Case I. $M_4 \leqslant M_2$.

In discussing this case we shall assume that $M_4 = 1$. Because of the linearity of our equation this is possible. We consider two possibilities. If $A_4M_3 \leqslant M_4/2$ and $b_3 \leqslant \ln 2$ we show that $y \in (4.1, 4)$ or there exists a modification z of y at $[3, 5]$. This however contradicts the fact that $y \in B_0(M, j)$ where $j = i(M, B_0)$ and so this possibility cannot occur. If $b_3 \geqslant \ln 2$ then we show that we may use Lemma 6.1 to decrease M_3 until $A_4M_3 > M_4/2$ or until $b_3 \leqslant \ln 2$. Thus in both cases we get that y or a modification z of y obtained by use of Lemma 6.1 satisfies condition (4.1) for $i = 4$.

We consider first the case $b_3 = \ln c \leqslant \ln 2$. If $M_3 < 1/\sqrt{2}$ then since $u_3 \geqslant 1/2$ we have $\int_0^{a_4} | y(x_3 + \sigma)\, d\sigma < \int_0^{\sqrt{2}} (1/\sqrt{2} - \sigma/2)\, d\sigma \leqslant 1/2$ and so $M_3A_4 > 1/2 = M_4/2$. We next note that $\varphi_2(y, -M_3) - \varphi_2(y, 1/c) = \ln c + A_4$. But if $y \notin B_3 \mid 4$ we have that $A_4M_3 < M_4/2$ and so $A_4 \leqslant 1/2M_3$. Thus the average slope of y for $x \in [\varphi_2(y, 1/c), \varphi_2(y, -M_3)]$ is

$$(M_3 + 1/c)/(\ln c + 1/2M_3) \geqslant M_3(M_3 + 1/2)/(M_3 \ln c + 1/2) > M_3 .$$

Thus $y'(x) < -M_3$ for $x \in [\varphi_2(y, M_2), \varphi_2(y, 1/c)]$ and it follows for $1/\sqrt{2} \leqslant M_3 \leqslant 1$ that $y'(x) \geqslant 1$ for $x \in [\varphi_3(y, -M_3), \varphi_3(y, 0)]$. For if $M_3 \geqslant 1/\sqrt{2}$ and $A_4M_3 \leqslant 1/2$ then $A_4 \leqslant 1/\sqrt{2}$, $a_3 + A_3 \geqslant \Delta_c - \ln 2 - 1/\sqrt{2}$, and so

$$(a_3 - 1/\sqrt{2}) + A_3 \geqslant 2 + 3/4 - \sqrt{2} > 1.$$

But then $\int_0^{a_4} | y(x_3 + \sigma)|\, d\sigma \leqslant M_3{}^2/2 < 1/2$ if $M_3 < 1$ which gives a contradiction. Thus we may assume that $M_3 \geqslant 1$. But then if $y \notin (4.1, 4)$ we have that $A_4 \leqslant 1/2$ and so $A_3 + a_3 \geqslant 2 + 1/4$. Since $y'(x) > M_3$ for $x \in [\varphi_2(y, M_2), \varphi_2(y, 1/c)]$ it then follows that $\int_0^{a_3} y(x_2 + \sigma)\, d\sigma + A_3M_2 \geqslant (3/8) + 7/4$ and so $M_3 \geqslant 2 + 1/8 - 1/2 = 13/8$ (here we have used $M_1 = 1$, $M_3 = 1$, $b_3 = \ln 2$, for larger values of M_1, M_3 or smaller values of b_3 the estimates for M_3 increase). Then replacing $M_3 \geqslant 1$ by $M_3 \geqslant 13/8$ we find by the above arguments that $M_3 \geqslant 13/8 + 5/26$. However in this case since $A_4M_3 \leqslant 1/2$ we have that $A_3 < 3$ and $\ln 2 + A_3 < 1$. Thus $\varphi_2(y, -M_3) - \varphi_2(y, 0) < 1$ and for any $x \in [\varphi_2(y, 0), \varphi_2(y, -M_3)]$ we have that $y(x)/[x - \varphi_2(y, 0)] < -M_3$ and for any $x \in [\varphi_2(y, M_2), \varphi_2(y, 0)]$ we have $y'(x) < -M_3$. Then if we define $z(x) = y(x)$ for $x \leqslant \varphi_2(y, M_2)$ and $M_{3+i}(z) = M_{5+i}(y)$ for $i \geqslant 0$ we will have that z is a modification of y at $[2, 4]$. This however contradicts the fact that $y \in B_0(M, j)$, $j = i(M, B_0)$. Thus if $y \in B_0(M, j)$, $j = i(M, B_0) > 4$, and $y \in (4.1, 3)$ then if $M_2(y) \geqslant M_4(y)$ and $u_3 \geqslant M_4/2$ it follows that $y \in (4.1, 4)$.

We now consider the case $b_3 = \ln c \geqslant \ln 2$. If $A_4M_3 \leqslant 1/2$ then $A_4 \leqslant 1/2M_3$ and $1/2 \leqslant \int_0^{a_4} | y(x_3 + \sigma)|\, d\sigma < M_3a_4$. Thus

$$a_4 > 1/2M_3 \qquad \text{and} \qquad M_3 - u_4 \geqslant M_2/2M_3 \qquad \text{for} \qquad M_3 \geqslant 1.$$

Denoting the left side of the inequality in Lemma 6.1 by $I(i+1)$ we then have for $M_3 \geqslant 1$ since $u_3 \leqslant 1/2$ and $M_2 \geqslant 1$ the estimate $I(3) \geqslant (1 + M_2/s_3) \cdot 1 \cdot 1/(2M_3)$. Thus we may apply Lemma 6.1 a finite number of times until the modified solution $z \in (4.1, 4)$, or $u_3 \geqslant 1/2$ which was treated earlier, or $M_3 \geqslant 1$. Now if $M_3 \leqslant 1$ then since $u_3 \leqslant 1/2$ we have that $s_3 = M_3 \leqslant 1$ and so $(1 + M_2/s_3) \geqslant 2$. Since $a_4 \geqslant 1/2$ and $y \in (4.1, 3)$ we have $I(3) \geqslant 2(1/u_3 - 1) M_3/2 \geqslant 1/2M_3$ if $M_3^2 \geqslant 1/2$. Here the first equality holds only if $M_3 = 1$ and the last only if $M_3^2 = 1/2$. Thus if $M_3 \in [1/\sqrt{2}, 1]$ we may apply Lemma 6.1 to reduce M_3. Since $y \in (4.1, 3)$ we have that $3M_3^2/8 + M_3x/4 \geqslant \int_0^{a_4} |y(x_3 + \sigma)| \, d\sigma \geqslant 1/2$ where $x = a_4 - M_3/2$. But for $M_3 \leqslant 1/\sqrt{2}$ this gives $x \geqslant 5/4M_3$. But then if s denotes the average slope of y for $x \in [\varphi_3(y, -M_3/2), \varphi_3(y, 0)]$ we have that $s \leqslant (M_3/2)/(1/x) \leqslant 2M_3^2/5$. Thus

$$x_4 - x_3 \geqslant 1 + \ln 2 + \ln 5/(4M_3^2) + M_3/2 + 5/(4M_3)$$

and for $M_3 \leqslant 1/\sqrt{2}$ this expression assumes its minimum value for $M_3 = 1/\sqrt{2}$ which exceeds Δ_c which contradicts $y \in B_0$. This completes the proof for the case in which $M_4 \leqslant M_2$.

Case II. $\qquad M_2 \leqslant M_4$.

As noted earlier we shall divide the discussion of this case into five subcases. In each of these cases we may assume without loss of generailty that $M_2 = 1$.

Case IIa. $\qquad M_3 \leqslant 1$.

We first note that $M_3 \geqslant 1/2$ for otherwise the average slope of y for $x \in [\varphi_3(y, -M_3), \varphi_3(y, 0)]$ would have to be less than $1/4(M_3)$ since $|\int_0^{a_4} y(x_3 + \sigma) \, d\sigma| > 1/2$ if $A_4 M_3 \leqslant M_4/2$. But then $x_4 - x_3 \geqslant 3 + 2 \ln 2$ which contradicts $y \in B_0$. We also note that since $y \in (4.1, 3)$ we may assume that $A_3 M_2 \geqslant M_3/2$. Otherwise $a_4 \leqslant A_3 \leqslant 1/2$ and

$$y(x_4 + A_4) - y(x_4 + A_4 + a_4) \leqslant 3/8 < M_4/2$$

which is impossible if $A_4 M_3 \leqslant M_4/2$. From the fact that $A_3 M_2 \geqslant M_3/2$ it follows that $M_4 - A_4 M_3 < |\int_0^{a_4} y(x_3 + \sigma) \, d\sigma| \leqslant 3/4$. To see this we note that if $b = \varphi_3(y, -M_3/2) - x_3$ then $|\int_0^b y(x_3 + \sigma) \, d\sigma| \leqslant 3M_3^2/8 \leqslant 3/8$. Then if $|\int_0^{a_4} y(x_3 + \sigma) \, d\sigma| \geqslant 3/4$ the average slope of y on $[x_3 + b, x_3 + a_4]$ must be less than $M_3/3$. This in turn would inply that

$$x_4 - x_3 \geqslant M_3/2 + 1 + 3/2 + \ln(3/M_3) > \Delta_c \quad \text{for} \quad 1/2 \leqslant M_3 \leqslant 1$$

which contradicts $y \in B_0$. Thus there exists $\epsilon > 0$ such that if $M_3 \leqslant 1$

then $A_4M_3 \geqslant M_4(1/4 + \epsilon)$. Also as a consequence of the fact that $A_3M_2 \geqslant M_3/2$ we have that $u_4 \leqslant M_3/4$ or $M_5 < u_4$ and $M_5 \leqslant M_3/4$. The last case is however ruled out by Lemma 6.4. If $M_5 \geqslant M_3/4$ then since $| \int_0^{a_4} y(x_3 + \sigma)\, d\sigma | \geqslant 1/8 + u_4$ it follows that $u_4 < M_3/4$ (i.e. for $u_4 = M_3/4$ the average slope of y for $x \in [\varphi_3(y, -M_3/2), \varphi_3(y, -M_3/4)]$ must be less than $M_3/4$ which would contradict $x_4 - x_3 \leqslant \Delta_c$).

We now have the necessary estimates to apply Lemma 6.1 to M_4 if $A_5 \leqslant 3/2$. For in this case we have

$$I(4) \geqslant (1 + M_3/M_4)(3/M_3)(M_4)(1/4 + \epsilon) > (M_4 + 1/M_3)\, 3/4 \geqslant 3/2.$$

If $A_5 \geqslant 3/2$ then since $M_4 - A_4M_3 < 3/4$ and $y \notin (4.1, 4)$ we must have that $M_5 > 3/4$. But since $u_4 \leqslant M_3/4$ we have that

$$\varphi_4(y, -M_5) - \varphi_4(y, -1/4) \geqslant \ln 2 + 1/4$$

and since $M_4 \geqslant 1$, $M_3 \leqslant 1$, we also have that

$$\varphi_4(y, -1/4) - \varphi_4(y, M_4) \geqslant 1 + 1/4.$$

Thus $3/2 \leqslant A_5 \leqslant \Delta_c - [2 + 1/2 + \ln 2] = 1 + 1/4$ a contradiction. Since our inequality is valid for all values of $M_4 \geqslant 1$ we may after a finite number of applications of Lemma 6.1 obtain a solution z for which $M_4(z) < M_2(z)$ (case 1) or $z \in (4.1, 4)$.

Case IIb. $\qquad M_3 \geqslant 1 \qquad$ and $\qquad u_3 \leqslant 1/2.$

In this case we will show that one may decrease M_3 by use of Lemma 6.1 until either $M_3(z) < 1$ (case IIa), or $z \in (4.1, 4)$, or $u_3(z) > 1/2$ and $1 \leqslant M_3(z) \leqslant 3/2$. The last possibility is considered in case IIc where we shall show that M_4 may be reduced until $M_4(z) \leqslant M_2(z)$ (case I), or $z \in (4.1, 4)$.

Thus we now proceed to obtain the necessary estimates needed to apply Lemma 6.1. Since $A_4M_3 \leqslant M_4/2$ we have that $a_4 > A_4$ and so $M_3 - u_4 \geqslant \min(A_4M_2, A_3M_2) = \min(A_4, A_3)$. We consider the two cases $s_3 > M_3$ and $s_3 = M_3$. If $s_3 > M_3$ then $x_3 - x_2 < 2$ and $A_3 > \Delta_c - 2 > 5/4$. Now if $A_4 \geqslant 1$ then $M_4 \geqslant 2M_3 \geqslant 2$ and since $u_3 \leqslant 1/2$, $M_2 = 1$, it follows that $x_4 - x_3 \geqslant 3/2 + 2 \ln 2$. But then $A_4 = \Delta_c - [x_4 - x_3] < 3/4$ and we have a contradiction. Thus $\min(M_3 - u_4, A_3M_2) = A_4$ and we have estimate $I(3) \geqslant (1 + 1/s_3)\, A_4 > A_4$ and we may apply Lemma 6.1. For the case $s_3 = M_3$ we note that if $A_3 \geqslant A_4$, then $\min(M_3 - u_4, A_3M_2) \geqslant A_4$ and we have the above estimate for $I(3)$. Thus we shall assume that $A_3 < A_4$. Now if $A_3 < M_3/2$ then since $y \in (4.1, 3)$ we have that $a_4 \leqslant A_3 < A_4$

which we saw earlier was impossible. Thus $A_3 \geqslant M_3/2$ and we have $I(3) \geqslant (1 + 1/M_3)\, A_3(1/u_3 - 1) \geqslant (M_3 + 1)/2 \geqslant 1$. But as we saw earlier $A_4 < 1$ (in fact $A_4 < 1 - \epsilon$ for some $\epsilon > 0$). Thus we may in this case also apply Lemma 6.1 as indicated. This completes the discussion of case IIb.

Case IIc. $\qquad 1 \leqslant M_3 \leqslant 3/2, \qquad u_3 \geqslant 1/2.$

In this case we shall show that we may always decrease M_4 or M_3 until $z \in (4.1, 4)$ or $M_4(z) \leqslant M_2(z)$ (case I). We first note that if h denotes the average slope of y for $x \in [x_2, \varphi_2(y, u_3)]$ then $h > 3/4$ and if $M_3 \geqslant 5/4$ then $h \geqslant 1$. For if $h \leqslant 3/4$ then

$$\varphi_3(y, 1/2) - \varphi_2(y, 1/2) \geqslant 1/(2h) + 1 + \ln(1/h) + 3/2 > \Delta_c$$

for $h \leqslant 3/4$ which contradicts $u_3 \geqslant 1/2$. For $M_3 \geqslant 5/4$ we have $\varphi_3(y, 1/2) - \varphi_2(y, 1/2) \geqslant 1/(2h) + 1 + \ln[5/(4h)] + 7/4 > \Delta_c$ for $h \leqslant 1$ which again contradicts $u_3 \geqslant 1/2$. Since we have that $h \leqslant M_1$ it follows that

$$\int_0^{a_3} y(x_2 + \sigma)\, d\sigma \leqslant \int_0^{(1-u_3)/h} (1 - h\sigma)\, d\sigma = (1 - u_3^2)/(2h) \leqslant 3/(8h) < 1/2 .$$

Since $u_3 \geqslant 1/2$ and $M_2 = 1$ this implies that $A_3 > M_3$.

We consider first the case $M_3 \in [1, 5/4]$. Then in order to reduce M_4 if we denote M_3A_4 by αM_4 it will suffice to show that

$$C = (1 + M_3/M_4)(1/u_4 - 1/M_3)\, \alpha M_4 > A_5 \qquad \text{for} \qquad \alpha \leqslant 1/2$$

since if $M_4 - u_5 \leqslant \alpha M_4$ we will have $y \in (4.1, 4)$. We first show that $A_5 < 23/10$. By Lemma 6.4 we may assume that either $M_5 > 1/4$ or $u_4 < 1/4$. Now if $M_5 \geqslant 1/4$ then $x_5 - x_4 \geqslant (1/2)(4/5) + a_4$. But if $u_4 \geqslant 1/4$ then

$$3/4 \leqslant \left| \int_0^{a_4} y(x_3 - \sigma)\, d\sigma \right| \leqslant \int_0^{a_4} (5/4 - \sigma)\, d\sigma = 5a_4/4 - a_4^2/2$$

and we have $a_4 \geqslant 1$ and $A_5 = \Delta_c - x_5 - x_4 < 23/10$ (we have considered here the case $\alpha = 1/2$, for $\alpha < 1/2$ the estimate for $x_5 - x_4$ increases). If $u_4 \leqslant 1/4$ then taking $M_3 = 5/4$, $M_4 = 1$ we have $a_4 > 1$ and so $\varphi_4(y, 0) - x_4 > (1/4)(4/5) + 1 > 6/5$. For $M_3 < 5/4$ and $M_4 \geqslant 1$ one may obtain the estimate $\varphi_4(y, 0) - x_4 > 6/5 + [(5/4)^2 - M_3^2]/8$. Thus in every case we have $A_5 < \Delta_c - 6/5 < 23/10$. We also note that $\alpha < 7/32$ since

$$(1 - \alpha) \leqslant (1 - \alpha)\, M_4 < \int_0^{M_3} (M_3 - \sigma)\, d\sigma \leqslant (5/4)^2(1/2) = 25/32.$$

We next note that

$$\partial C/\partial M_4 = (1/u_4 - 1/M_3)\,\alpha - (M_4 + M_3)\,\alpha \frac{\partial u_4}{\partial M_4}\Big/ u_4^2 > 0,$$

since for M_3 and α fixed we have $\partial u_4/\partial M_4 \leqslant 0$. Thus it will suffice to establish our desired inequality for $M_4 = 1$ and α, M_1 in the allowable range. For $7/32 \leqslant \alpha \leqslant 11/32$ we have that $u_4 + 21/32 \leqslant \int_0^{M_3 - u_4} (M_3 - \sigma)\, d\sigma = (M_3^2 - u_4^2)/2$ and so $u_4 < 1/8$ and $C > 2 \cdot 7 \cdot (7/32) > 3$. For $11/32 \leqslant \alpha \leqslant 13/32$ and $13/32 \leqslant \alpha \leqslant 1/2$ one obtains by the above procedure the estimates $u_4 \leqslant 3/16$, $C > 11/4$, and $u_4 \leqslant 1/4$, $C > 39/16 > 24/10$. Thus in case we have $C > 24/10 > 23/10 > A_5$ and we may apply Lemma 6.1 a finite number of times to obtain $z \in (4.1, 4)$ or $M_4(z) \leqslant M_2(z)$ (case I).

We now consider the case where $M_3 \in [5/4, 3/2]$. We first note that if $M_4 \geqslant M_3$ then we may decrease M_4. Since for $M_3 \leqslant 3/2$ we have $\int_0^{3M_3/4} (M_3 - \sigma)\, d\sigma < 3M_3/4$ it follows from Lemma 6.4 that $u_4 \leqslant M_5$ and so

$$u_4 = \int_0^{M_3 - u_4} (M_3 - \sigma)\, d\sigma - M_4(1 - \alpha) < M_3^2/2 - M_3 + M_3\alpha.$$

Thus $(M_4 + M_3)(1/u_4 - 1/M_3)\,\alpha > (5/2)(2/3)(1/\alpha - 1/4) - 1)\,\alpha$ which assumes its minimum value $2 + 1/2$ for $\alpha = 1/2$. But since $M_4 \geqslant M_3$ we have that $A_5 < \Delta_e - 1 < 2\,1/2$ and we may decrease M_4.

It remains to consider for $M_3 \in [5/4, 3/2]$ the case in which $M_4 \leqslant M_3$. We consider first the case $u_3 \in [1/2, 3/4]$. It again follows from Lemma 6.4 that $M_5 \geqslant M_4/4$ or $u_4 \leqslant M_5$. However if $x = 9M_4/(10M_3)$ then $\int_0^x (M_3 - \sigma)\, d\sigma < 3M_4/4$. Thus if $A_4 M_3 \leqslant M_4/2$ we have that $M_3 - u_4 > x$ and

$$\begin{aligned}(1 + M_4/M_3)(1/u_3 - 1/M_4)(M_3 - u_4) &> (5/3)(1/3)(9M_4/10M_3)\\ &= M_4/2M_3 \geqslant A_3.\end{aligned}$$

Thus after a finite number of applications of Lemma 6.4 we will have $u_3(z) > 3/4$ or $M_3(z) \leqslant M_4(z)$ which was considered earlier.

We now show that if $u_3 \geqslant 3/4$ then we may decrease M_4. We first note that since $M_3 \geqslant 5/4$, $u_3 \geqslant 3/4$ we have that $\varphi_3(y, 3/4) - x_3 \leqslant M_3 + 3/4 + 1/32$ and in general $\varphi_3(y, 3/4 - \sigma) - x_3 < M_3 + 3/4 + 1/32 - \sigma$ for any $0 < \sigma \leqslant 3/4$. Thus if $M_4 \leqslant 2(3/4 - 1/32)$ we have

$$\varphi_3(y, M_4) - x_3 < 2 + 1/4 + \ln 2 \qquad \text{and so} \qquad A_2 > 1/2.$$

Since $M_4 \leqslant M_3$ we would then have $y \in (4.1, 4)$. Thus we have that $2\,3/16 < M_4 \leqslant M_3 \leqslant 3/2$. Since $M_4 \leqslant 3/2$ we also have that $A_4 \geqslant 15/32$. Since $A_3 \geqslant M_3$ we have that $\int_0^{13/12} (M_3 - \sigma)\, d\sigma \leqslant 299/288 < 5/12 + 23/32$ and so by Lemma 6.4 we have that $u_4 < 5/12$. Thus if $A_5 \leqslant 3/2$ we have

$(1 + M_3/M_4)(1/u_4 - 1/M_3) A_4 M_3 > 2((12/5)(5/4) - 1) A_4 > 7/4 > A_5$. If $A_5 \geqslant 3/2$ then since $M_4 - M_3 A_4 \leqslant 3/2 - 75/128 < 1$ and $M_4 A_5 \geqslant 69/32$ we have that $M_5 > 37/32$ (otherwise $a_5 = 0$ and $y \in (4.1, 4)$. But then $x_5 - x_4 > (M_4 + M_5)/M_3 \geqslant 1 + 3/4$ and so $A_5 = \Delta_c - (x_5 - x_4) < 1 + 3/4$. We may now apply Lemma 6.1 to decrease M_4. Thus after a finite number of applications of Lemma 6.1 we have that our modified solution $z \in (4.1, 4)$ or $M_4(z) \leqslant M_2(z)$ and we are in case I.

Case IId. $\qquad M_3 \geqslant 3/2, \qquad 2/3 \leqslant u_3 \leqslant 1.$

We first note that if $A_3 \geqslant 2 + 1/4$ and $M_3 \geqslant 3/2$ then we may apply Lemma 6.2 to increase M_2. For in this case we have

$$A_2 \leqslant A_2 + a_2 \leqslant \Delta_c - a/4 - \ln(3/(2u_2)) = 1/2 + \ln(4/(3u_2)).$$

Thus we consider the expression

$$(1 + 1/2 + \ln(4/3u_2))/u_2 \geqslant (1 + A_2) u_3/u_2 .$$

The maximum value of the left side is obtained for $u_2 = 3e^{-1/2}/4$ and is less than 2.2. Thus we have $(1 + A_2) u_3/u_2 < 2.2 < 9/4 \leqslant A_3$ and we may apply Lemma 6.2 to increase M_2.

We consider first the case $u_3 \geqslant .79$. We first note that we may restrict ourselves to the case $u_2 \geqslant 3/5$. For if we let

$$\begin{aligned} h = [u_2 + u_3]/[\varphi_2(y, -u_2) - \varphi_2(y, u_3)] &\geqslant (.79 + u_2)/[\Delta_c - (M_3 + .79)] \\ &\geqslant [.79 + u_2]/[.46 + \ln(4u_2/3)] \end{aligned}$$

then for $u_2 \leqslant 3/5$ we have $h > 5.3$. But then $a_3 \leqslant .21/h < .4$ and so $A_3 > M_3 + 3/4 \geqslant 2 + 1/4$ and by the previous result we may increase M_2. If $u_2 \in [3/5, 3/4]$ then we have

$$h > 3.34 \qquad \text{and} \qquad A_3 > 1.5 + (.79 - .063) = 2.227.$$

We also have that $(1 + A_2)/u_2 \leqslant (1.523 + \ln[(4u_2)/3])/u_2 < 2.17$. Thus for $u_2 \in [3/5, 3/4]$ we may use Lemma 6.2 to increase M_2. By similar reasoning one finds for $u_2 \in [3/4, 1]$ that $h > 2.38$, $A_3 > 2.20$, and $(1 + A_2)/u_2 \leqslant (1 + .55 + \ln(4u_2/3))/u_2 < 2.068$. For $u_2 \in [1, 3/2]$ one has $h > 1.97$, $A_3 > 2.183$ and $(1 + A_2)/u_2 \leqslant (1 + .57 + \ln(4u_2/3)) < 1.87$. Thus in both these cases we may increase M_2. Finally we note that if $u_2 \geqslant 3/2$ and $u_3 \geqslant 1/2$ then $A_3 \geqslant M_3 + u_3 - (1 - u_3) 2/3$ and

$$u_3(1 + A_2)/u_2 \leqslant 2u_3(1 + 2M_3/3 + 2/3)/3 < M_3 + 5u_3/3 - 2/3 - 1/6.$$

Thus if $u_3 \geqslant .79$ or $u_2 \geqslant 3/2$ and $u_3 \geqslant 1/2$ then we may use Lemma 6.2 to increase M_2.

We now consider the case in which $2/3 \leqslant u_3 \leqslant .79 < 4/5$ and again show that Lemma 6.2 may be used to increase M_2. As noted above we may restrict ourselves to the case in which $u_2 \leqslant 3/2$. As before we obtain estimates for A_3 and $u_3(1 + A_2)/u_2$. We again let

$$h = [u_3 + u_2]/[\varphi_2(y, -u_2) - \varphi_2(y, u_3)] \geqslant [2/3 + u_2]/[7/12 + \ln(4u_2/3].$$

Then if $u_2 \leqslant 3/5$ we have $h > 2$, $a_3 < 1/6$, and $A_3 > 1 + 1/2 + 1/2 = 2$. In fact if we consider A_3 as a function of $u_3 A_3(u_3) \geqslant 2 + (u_3 - 2/3) = \alpha(u_3)$ for $[u_3 \in 2/3, 4/5]$. We also have $A_2(u_3) \leqslant \Delta - A_3(u_3) - \ln 3/(2u_2)$. Thus we shall show that

$$M(u_3) = \max_{u_2 \leqslant 3/5} \{[1 + A_2(u_3)]\, u_3/u_2\} \leqslant \max_{u_2 \leqslant 3/5} \{[7/4 + \ln(4u_2/3) - (u_3 - 2/3)]\}$$
$$= \sigma(u_3) < A_1(u_3)$$

for $u_3 \in [2/3, 4/5]$. However, an easy computation for u_3 fixed gives that the maximum in the expression defining $\sigma(u_3)$ is assumed for $u_2 = 3\exp[-3/4 + (u_3 - 2/3)]/4$ and is $u_3 4\exp[+3/4 - (u_3 - 2/3)]/3$. However, since $d\sigma(u_3)/du_3 \leqslant 1$ and $d\alpha(u_3)/du_3 = 1$ for $u_3 \in [2/3, 4/5]$ our desired inequality follows from the fact that for $u_2 \leqslant 3/5$

$$A_3(2/3) \geqslant 2 > 1.89 > \sigma(2/3) \geqslant M(2/3).$$

We next consider $u_2 \in [3/5, 3/4]$. For $u_3 \in [2/3, 3/4]$ we have $h \geqslant 17/7$ and $A_3 > 1.5 + 2/3 - (1/3)(17/7) > 2.02$. For $u_3(1 + A_2)/u_2$ we have the upper bound

$$3(1 + 7/5) + 7/12 + \ln(4u_2/3))/u_2 < 1.9$$

For $u_3 \in [3/4, 4/5]$ we have $h \geqslant 17/7$, $A_3 > 2.11$ and $u_3(1 + A_2)/u_2 < 2$. For $u_2 \in [3/4, 1]$, $u_3 \in [2/3, 4/5]$, we have $h > 1.8$, $A_3 > 1.96$ and $u_3(1 + A_2)/u_2 < 1.92$. For $u_2 \in [1, 3/2]$, $u_3 \in [2/3, 4/5]$ we have $h > 13/8$, $a_3 < .21$, $A_3 > 1.95$ and $u_3(1 + A_2)/u_2 \leqslant 1.68$. Thus in all these cases we may apply Lemma 6.2 to increase M_2 until $M_2 \geqslant M_4$ (case I) or Condition (4.1) is satisfied for $i = 4$.

Case IIe. $\qquad M_3 \geqslant 3/2, \qquad u_3 \in [1/2, 2/3].$

We first note that if $M_3 \geqslant 2$ then since $u_3 \geqslant 1/2$ we have that h as defined in case IId exceeds $2 + 1/2$ and so $A_3 > 9/4$. But as seen in case IId then M_2 may be increased. Thus we shall assume that $M_3 \leqslant 2$ and as noted in case IId we may assume that $s_3 = M_3$. Thus in order to apply Lemma 6.1 to decrease M_3 it will suffice to show that $(1 + 1/M_3)(1/u_3 - 1)\, a_4 \geqslant 3a_4/4 > A_4$. However if $|\, y(x_4 + A_4 + a_4)| \geqslant 3M_4/7$ then we have that

$$(1/2 + 3/7)\, M_4 \leqslant \left| \int_0^{a_4} y(x_3 + \sigma)\, d\sigma \right| \leqslant a_4 M_3$$

and $a_4 \geqslant 13M_4/(14M_3) \geqslant 13A_4/7 > 4A_4/3$. Thus we may assume that $a_4 = M_3 - u_4$ and $u_4 < 3M_4/7$. Now if $M_4 \leqslant M_3$ then

$$a_4 \geqslant M_3 - 3/7M_4 \geqslant 4M_3/7 > 2/3 \geqslant 4M_4/(3 \cdot 2M_3) \geqslant 4/3A_4 .$$

If $M_4 \geqslant M_3$ then since $M_3 \geqslant 3/2$ we have $x_4 - x_3 \geqslant 2 + 1/2 + \ln 3/2$ and so $A_4 < 3/5$. Thus it will suffice to show that $a_4 > 4/5$. Since

$$M_3 - a_4 = u_3 = M_3 a_4 - a_4{}^2/2 - M_4/c,$$

where $c \leqslant 2$ we have $a_4 \geqslant 3M_3/[2(M_3 + 1 - a_4/2)]$. Since $M_1 \geqslant 3/2$, $a_4 \leqslant M_3 \leqslant 2$ this expression assumes its minimum value at $M_3 = 3/2$ and we have $a_4 \geqslant 9/10 > 4/5$. Thus in every case we may either decrease M_3 until $z \in (4.1, 4)$ or we are in an earlier case, or we may increase M_2 until condition (4.1) is satisfied or we are in an earlier case.

This completes the proof of the induction step except to observe that the cases in which it was necessary to apply Lemma 6.2 can not arise if $2 \leqslant l \leqslant 3$. We first note that if $M_1 \leqslant .695$ then $A_1 M_0 = A_1 > 1.7$. This is clear since $A_0 = 3/4 + \ln 2 > 1.44$ and letting $a_0 = .3$ one has that $y(A_0 + a_0) < -.695$. Then if $b_0 = 0$ one has that $A_1 = 1.7$. Thus we shall assume that $M_1 > .69$. Now if $M_1 \geqslant 3/2$ then $x_1 - x_0 > 7/4 + \ln 2$, $A_1 < 1$, $\int_0^{a_1} y(\sigma)\, d\sigma \leqslant 1/2$ and we have a contradiction. Thus $M_1 < 3/2M_0$ and so cases IId and IIe can not arise for $l = 2$. We next note that if $M_1 \leqslant 3/2$ then $A_1 > .87$. But then if $1 \leqslant M_1 \leqslant 3/2$ and $M_2 \geqslant 3M_1/2 \geqslant 3/2$ we have setting $M_1 = 1 + \sigma$, that $x_2 - x_1 > \sigma + 7/4 + \ln 2$ and so $A_2 < 1 - \sigma$, $A_2 M_1 < 1 < M_1$. Thus $\int_0^{a_2} |\, y(x_1 + \sigma)|\, d\sigma > M_1$ if $u_2 \geqslant M_1/2$. But $\int_0^{a_2} |\, y(x_1 + \sigma)|\, d\sigma > M_1$ if $u_2 \geqslant M_1/2$. But $\int_0^{a_2} |\, y(x_1 + \sigma)|\, d\sigma > M_1$ implies that $a_1 > 1$ which in turn implies that $u_2 < M_1/2$. Thus if $M_1 \in [1, 3/2]$ cases IId, IIe can not arise for $l = 3$. If $M_1 \in [.69, 1]$ then $u_1 \leqslant 1 < 3M_1/2$. We also note that if $u_2 \geqslant 2M_1/3$ then $a_2 \leqslant M_1/3$ and $M_1 A_2 > M_2 + M_1/3$ or $A_2 > 1.83$. But $A_2 = \Delta_c - (x_2 - x_1) < 3.5 - 1.7 = 1.8$ and so $u_2 < 2M_1/3$. Finally we note that if $M_2 \geqslant 2M_1$ then $A_2 < 3/2$ and $u_2 < M_1/2$. Thus the cases in which it was necessary to use Lemma 6.2 can not arise for $l = 3$. This completes the induction step and the proof of Theorem 6.3.

THEOREM 6.5. *If $\Delta = \Delta_c$ then every oscillatory solution of the unstable equation* (1.2) *is bounded.*

Proof. Assume (1.2) has an unbounded oscillatory solution. Then by Theorem 2.4 there is a solution $y \in B_1$, $y(\infty) = \infty$. It then follows from Theorem 5.3 that there is a solution $y \in B_2$, $y(\infty) = \infty$. Then setting $M = \Delta_c^9 \exp(2\Delta_c)$ in Theorem 6.3 let $z \in B_3 \mid i - 1$, $M_i(z) \geqslant M$. Then $M_2(z) \leqslant \Delta_c^2$, $M_3(z) \leqslant \Delta_c^3$ and by Corollary 4.2, $M_j(z) \leqslant \Delta_c^5 \exp(2\Delta_c)$ for $j \leqslant i - 3$. However, since $z \in B_0$, we have that $M_{j+3}(z) \leqslant \Delta_c^3 M_j$ for every $j \geqslant 0$. Thus we have that $M \leqslant M_i \leqslant \Delta_c^3 M_{i-3} \leqslant \Delta^{-1} M$ and so it follows that (1.2) for $\Delta = \Delta_c$ may not have unbounded oscillatory solutions.

Finally we note that using Theorem 6.3 one could construct an unbounded solution $w \in B_3$ if there exists $y \in B_2$, $y(\infty) = \infty$. Thus the result in Theorem 6.5 would follow from Theorem 4.1.

THEOREM 6.6. *If there exists $y \in B_2$, $y(\infty) = \infty$, then there exists $w \in B_3$, $w(\infty) = \infty$.*

Proof. In Theorem 6.3 select $M > 8\Delta_c^5$ and let $z \in B_3 \mid i - 1$ be such that $M_i(z) \geqslant M$. Then if $M_{i-3} \geqslant M_{i-4}$ we have that either $A_{i-3} M_{i-4} \geqslant M_{i-3}/2$ or $A_{i-4} M_{i-5} \geqslant M_{i-4}/2$. In any case there exists a $j \leqslant i - 3$ such that $M_j \geqslant M/\Delta_c^4$ and $A_j M_{j-1} \geqslant M_j/2$. Now for $x \leqslant \varphi_{j-1}(z, 2\operatorname{sgn}[z(x_j)])$ we let $w(x) = z(x)$. We then set $M_j(w) = 2$, $M_{j+1}(w) = 2$, and $M_{j+2}(w) = 2$. Since $M_{j-1}(z) > 8$ and $M_j(z) > 8\Delta_c$ we have that $x_{j+1}(w) - x_j(w) < 1/2$ and $| w'(x) | = 2$ for $x \in [x_{j+1}(w), x_{j+2}(w)]$. We now define

$$w(x) = ((-1)^{j+2}\, 2)^k\, w(x - k x_{j+2}(w))$$
$$\text{for} \quad x \in [k x_{j+2}(w), (k+1)\, x_{j+2}(w)] \qquad \text{and} \qquad k \geqslant 1.$$

It is clear that $w \in B_3$ and $w(\infty) = \infty$.

REFERENCES

1. BELLMAN, R. AND COOKE, K. L., "Differential-Difference Equations." Academic Press, New York, 1963.
2. BUCHANAN, J., On the Stability of Solutions of a Differential-Difference Equation with a Variable Lag. Ph.D. Thesis, Purdue University, 1964.
3. MYSKIS, A. D., "Linear Differential Equations with Retarded Arguments," (German). Deutscher Verlag Der Wissenschaften, Berlin, 1955.